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ON SOME PROJECTION METHODS FOR THE SOLUTION OF NONLINEAR EQUATIONS WITH NONDIFFERENTIABLE OPERATORS

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Abstract. We consider a nonlinear equation with a nondifferentiable operator in a Banach space. We approximate a solution of the nonlinear equation using an iteration, whose iterates can be obtained by solving a certain operator equation in a finite dimensional space.

1. Introduction

We study the problem of approximating a fixed point x^* of the equation

$$x = F(x), \tag{1}$$

in a Banach space E, where F is a nondifferentiable continuous operator defined on some convex subset $D \subset E$ with values in E. Let F_1 be another continuous operator defined on E with values in E, and let P be a linear projection operator ($P = P^2$) which projects E on its subspace E_P and set $Q = I - P$. We will assume that the operator PF_1 is Fréchet differentiable on $D \subset E$.

We will approximate a fixed point x^* of equation (1), using the approximations

$$x_{n+1} = F(x_n) + PF_1'(x_n)(x_{n+1} - x_n), \quad n = 0, 1, 2, \ldots \tag{2}$$

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for some $x_0 \in D$, where $PF_1'(x_n)$ is the Fréchet derivative of $PF_1(x)$ at x_n. The iteration (2) has been considered by many authors when $P = I$, the identity operator [4], [7], [8], [9], or when $P = I$ and $F = F_1$ [1], [2], [3], [5], [6].

In this paper we assume that for $x_0 \in D$ the inverse $(I - PF_1'(x_0))^{-1}$ exists and for $\lambda \in (0,1]$, the following Hölder-continuity assumptions are satisfied

$$\|(I - PF_1'(x_0))^{-1}[PF_1'(x) - PF_1'(y)]\| \le c_1\|x - y\|^\lambda, \tag{3}$$

$$\|(I - PF_1'(x_0))^{-1}[QF_1(x) - QF_1(y)]\| \le c_2\|x - y\| \tag{4}$$

and

$$\|(I - PF_1'(x_0))^{-1}[G(x) - G(y)]\| \le c_3\|x - y\|, \; G(x) = F(x) - F_1(x) \tag{5}$$

for all $x, y \in D$ and some c_1, c_2 and $c_3 \ge 0$.

Note that in case of convergence, the iteration (2) converges to a fixed point x^* of equation (1). Moreover, the problem of computing the iterates $\{x_n\}$, $n \ge 0$ is equivalent to solving a system of linear algebraic equations of order at most N, where $N = \dim(E_p)$, if E_P is finite dimensional.

We finally apply our results to solve an integral equation with a nondifferentiable kernel.

2. Existence-Uniqueness Theorems

We now provide some sufficient conditions for the convergence of iteration (2) to fixed point of equation (1).

Theorem 1. *Assume that the following conditions hold:*

(i) *for $x_0 \in D$ the linear operator $(I - PF_1'(x_0))^{-1}$ exists and*

$$\|[I - PF_1'(x_0)]^{-1}(x_0 - F(x_0))\| \le \eta; \tag{6}$$

(ii) *the operators $PF_1'(x)$, $QF_1(x)$ and $G(x)$ satisfy the conditions (3)-(5) respectively, for some $\lambda \in (0,1]$;*

(iii) *the ball* $\overline{U}(x_0, R_0) = \{x \in E| \ \|x - x_0\| \leq R_0\} \subset D$ *where*

$$\eta \frac{h}{1-h} \ \leq \ R_0, \tag{7}$$

$$h \ = \ \frac{bc_1}{1+\lambda}\eta^\lambda + c, \quad c = b(c_2 + c_3), \tag{8}$$

and

(iv) *the quantities* h, b, c_1, c *and* R_0 *satisfy*

$$h < 1, \ c_1 R_0^\lambda < 1 \text{ and } b \geq \frac{1}{1 - c_1 R_0^\lambda}. \tag{9}$$

Then, the iterates generated by (2) *are well defined for all* $n \geq 0$, *remain in* $\overline{U}(x_0, R_0)$ *and converge to a fixed point* $x^* \in \overline{U}(x_0, R_0)$ *of equation* (1), *with*

$$\|x_n - x^*\| \leq \eta \frac{h^n}{1-h}, \quad n = 0, 1, 2, \ldots. \tag{10}$$

Proof. Using the Banach lemma on invertible operators, (9), (3) and (2) we obtain that the linear operator $I - PF_1'(x)$ is invertible of $\overline{U}(x_0, R_0)$ and

$$\|(I - PF_1'(x))^{-1}(I - PF_1'(x_0))\| \leq b. \tag{11}$$

By (2) we get

$$\begin{aligned} e_{n+1} \ &= \|x_{n+1} - x_n\| \leq \|[I - PF_1'(x_n)]^{-1}(I - PF_1'(x_0))\| \\ &[\|(I - PF_1'(x_0))^{-1}\{PF_1(x_n) - PF_1(x_{n-1}) - PF_1'(x_{n-1})(x_n - x_{n-1})\}\| \\ &+ \|(I - PF_1'(x_0))^{-1}(QF_1(x_n) - QF_1(x_{n-1}))\| \\ &+ \|(I - PF_1'(x_0))^{-1}(G(x_n) - G(x_{n-1}))\|]. \end{aligned} \tag{12}$$

Using (3)-(5), (11), (12) and the finite difference formula, we get

$$e_{n+1} \leq \frac{bc_1}{1+\lambda} e_n^{1+\lambda} + bc_2 e_n + bc_3 e_n = \left[\frac{bc_1}{1+\lambda} e_n^\lambda + c\right] e_n. \tag{13}$$

We will now show using induction on n that

$$e_{n+1} \leq \eta h^{n+1}, \quad n = 0, 1, 2, \ldots. \tag{14}$$

From (13) for $n = 0$, we get using (6) and (8) that $e_1 \leq \eta h$. Hence, inequality (14) is true for $n = 1$.

Let us assume that inequality (14) is true for all $k \leq n$. Then

$$e_{k+1} \leq \left[\frac{bc_1}{1+\lambda} e_k^\lambda + c \right] e_k \leq h\eta h^k \leq \eta h^{k+1}.$$

That is, (14) is true for all $n = 0, 1, 2, \ldots$.

We now assume that $x_j \in \overline{U}(x_0, R_0)$ for $j = 0, 1, 2, \ldots, n$. Then, we get

$$\|x_0 - x_{n+1}\| \sum_{j=1}^{n+1} e_j \leq \eta \sum_{j=1}^{n+1} h^j = \eta h \frac{1 - h^{k+1}}{1 - h} \leq R_0.$$

Hence, $x_{n+1} \in \overline{U}(x_0, R_0)$.

Moreover,

$$\|x_n - x_{n+k}\| \leq \sum_{j=0}^{k} e_{n+j} \leq \eta h^n \frac{1 - h^{k+1}}{1 - h} \leq \eta \frac{h^n}{1 - h}. \tag{15}$$

That is, the sequence $\{x_n\}$ is Cauchy in a Banach space and as such it converges to some $x^* \in \overline{U}(x_0, R_0)$. By taking the limit in (15) we obtain (10). Furthermore, by taking the limit in (2) we obtain that $x^* = F(x^*)$.

That completes the proof of the theorem.

We can now prove the following theorem.

Theorem 2. *Assume that the following are true:*

(i) *equation* (1) *has a fixed point* $x^* \in \overline{U}(x_0, R_1)$;

(ii) *the hypotheses* (*i*) *and* (*ii*) *of Theorem* 1 *are true;*

(iii) *the ball* $\overline{U}(x_0, R_2) \subset D$ *with*

$$R_2 \geq (1 + h_1) R_1,$$

where

$$h_1 = \frac{b_1 c_1}{1 + \lambda} R_1^\lambda + c < 1, \quad c_1 R_1^\lambda < 1$$

and

$$b_1 \geq \frac{1}{1 - c_1 R_1^\lambda}.$$

3. Applications

Consider the integral equation of the form

$$x(t) = \int_0^1 K(t,s,x(s))ds$$

in the space $E = C[0,1]$, where $K(t,s,x(s))$ is nondifferentiable on some $S \subset E$. Set

$$F(x) = \int_0^1 K(t,s,x(s))ds \text{ and } F_1(x) = \int_0^1 L(t,s,x(s))ds,$$

where $L(t,s,x(s))$ is a differentiable operator on D. Then

$$PF_1'(x) = \int_0^1 \overline{L}_x'(t,s,x(s))ds,$$

where

$$\overline{L}(t,s,x(s)) = \sum_{i=1}^m M_i(t)N_i(s,x(s))$$

is a degenerate kernel approximating the function $L(t,s,x)$ on S. The operator $\overline{L}$ can be a portion of the Taylor or Fourier series for the operator $L(t,s,x)$ if we consider it as a function of t. The iteration (2) can now be written as

$$\begin{aligned} x_{n+1}(t) &= \int_0^1 K(t,s,x_n(s))ds - \int_0^1 \overline{L}_x'(t,s,x_n(s))x_n(s)ds, \\ &+ \int_0^1 \overline{L}_x'(t,s,x_n(x))x_{n+1}(s)ds. \end{aligned}$$

Set,

$$f_n(t) = \int_0^1 K(t,s,x_n(s))ds - \int_0^1 \overline{L}_x'(t,s,x_n(s))x_n(s)ds,$$

then iteration (17) becomes

$$x_{n+1}(t) = f_n(t) + \sum_{i=1}^m M_i(t) \int_0^1 N_i'(s,x_n(s))x_{n+1}(s)ds,$$

which can be solved to give a family of equations

$$c_j - \sum_{i=1}^m a_{ji}c_i = b_j, \quad j = 1,\ldots,m,$$

where

$$c_j = \int_0^1 N_j'(s, x_n(s))x_{n+1}(s)ds, \quad j = 1, \dots, m,$$

$$a_{ji} = \int_0^1 N_j'(s, x_n(s))M_i(s)ds, \quad j = 1, \dots, m, \ i = 1, \dots, m,$$

and

$$b_j = \int_0^1 N_j'(s, x_n(s))f_n(s)ds, \quad j = 1, \dots, m.$$

The above family is a system of linear algebraic equation. If the determinant $D(x_n)$ of this system is not equal to zero, then

$$c_i = \frac{1}{D(x_n)} \sum_{k=1}^{m} D_{ki}(x_n)b_k$$

and

$$x_{n+1}(t) = f_n(t) + \sum_{i=1}^{m}\sum_{k=1}^{m} \frac{D_{ki}(x_n)a_{ki}}{D(x_n)},$$

where $D_{ki}(x_n)$ is the cofactor of the element in the i-th row and k-th column of the determinant $D(x_n)$.

Let us suppose now that the operators $\overline{L}_x'(t,s,x)$, $Q(t,s,x)$, $G(t,s,x)$ and $T(t,s,x)$, where $Q(t,s,x) = L(t,s,x) - \overline{L}(t,s,x)$, $G(t,s,x) = K(t,s,x) - L(t,s,x)$, and

$$T(s,x,x) = \frac{1}{D(x)} \sum_{i=1}^{m}\sum_{k=1}^{m} M_i(t)D_{ki}(x)N_k'(s,x),$$

satisfy the conditions

$$|\overline{L}_x'(t,s,x) - \overline{L}'(t,s,y)| \le c_1(t,s)|x-y|^{\lambda},$$

$$|Q(t,s,x) - Q(t,s,y)| \le c_2(t,s)|x-y|,$$

$$|G(t,s,x) - G(t,s,y)| \le c_3(t,s)|x-y|,$$

and

$$|T(t,s,x)| \le r_1(t,s) \text{ on } S.$$

Then the constants appearing in Theorem 1 can be estimated as follows:

$$c_1 \sup_{t\in[0,1]} \int_0^1 c_1(t,s)ds, \quad c_2 \le \sup_{t\in[0,1]z} \int_0^1 c_2(t,s)ds,$$

and

$$c_3 \leq \sup_{t\in[0,1]} \int_0^1 c_3(t,s)ds \text{ and } b \leq 1 + \sup_{t\in[0,1]} \int_0^1 r_1(t,s)ds.$$

Once this is achieved on a specific example, we then define c_1, c_2, c_3, and b to be equal to the quantities appearing at the right hand side of the above inequalities respectively. Then we try various guesses for the starting point x_0 until we find one that together with the rest of the parameters satisfy the hypotheses (i)-(iv) of Theorem 1. Theorem 1 can then be applied to solve the integral equation.

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