TAMKANG JOURNAL OF MATHEMATICS
Volume 24, Number 1, Spring 1993

REMARKS ON THE *-TOPOLOGY

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Abstract. An ideal I on a set X is a collection of subsets of X which is closed under the operations of subset (heredity) and finite union (additivity). Ideals are useful in generation new spaces from the old ones. The central theme in this paper is to give new characterizations and properties to the *-topology in the sense of Hashimoto or I-topology in the sense of Vaidyanathaswamy and $\tau^*(I)$ in the sense of Hamlett, Rose and Janković. Several connections between the *-topology and other corresponding ones are investigated.

1. Introduction

One type of topology via ideals has been defined by three independent authors. In 1945, Vaidyanathaswamy [33] called it I-topology. On the other hand in 1976, Hashimoto [12] named it the *-topology. Recently in 1990, Hamlett, Rose and Janković [14] and [11] called it $\tau^*(I)$.

The purpose of the present paper is to investigate further characterizations, properties and some connections of the topology $\tau^*(I)$ with other corresponding ones. Also, in section 4, we study the topology $\tau^*(I_n)$, where I_n denotes the ideal of nowhere dense subsets.

2. Preliminaries

Throughout the present paper, spaces mean topological spaces on which no separation axiom is assumed unless explicitly stated. Let A be a subset of a space

Received September 30, 1991.

(X,τ). We denote the closure of A and the interior of A with respect to τ by $Cl(A)$ and Int(A). We denote the open nbd system for a point x in a space (X,τ) by $N(x)$, i.e., $N(x) = \{U \in \tau : x \in U\}$. $P(A)$, the power set of A,. A subset A of a space (X,τ) is said to be regular open (resp. regular closed) if Int $(Cl(A)) = A$ (resp. $Cl(\text{Int}(A)) = A$). A subset A of (X,τ) is said to be α-open [26] (resp. semi-open [16], preopen [20], β-open [1]) if $\text{Int}(Cl(\text{Int}(A))) \supset A$ (resp. $Cl(\text{Int}(A)) \supset A$, $\text{Int}(Cl(A)) \supset A$, $Cl(\text{Int}(Cl(A))) \supset A$) The complement of semi-open (resp. preopen) is called semi-closed (resp. preclosed). The family of all regular open (resp. regular closed, α-open, semi-open, semi closed, preopen, preclosed, β-open) sets of (X,τ) is denoted by $RO\,(X,\tau)$ (resp. $RC\,(X,\tau)$, $\alpha O(X,\tau)$ or τ^α, $SO\,(X,\tau)$, $SC\,(X,\tau)$, $PO\,(X,\tau)$, $PC\,(X,\tau)$, $\beta O\,(X,\tau)$). It was observed in [26] that τ^α is a topology on X and that $SO\,(X,\tau) \supset \tau^\alpha \supset \tau$. Recall that a subset A of (X,τ), is said to be nowhere dense if Int $Cl(A) = \Phi$, and is called Co-dense in X if $X - A$ is dense or Int $(A) = \Phi$.

Given a nonempty set X, an ideal I [15] is defined to be a nonempty collection of subsets of X such that:

(1) $B \in I$ and $B \supseteq A \to A \in I$ (heredity), and

(2) $A \in I$ and $B \in I \to A \cup B \in I$ (finite additivity) if, in addition, I satisfies the following condition:

(3) $\{A_n : n = 1, 2, \cdots\cdots\} \subseteq I \to \cup A_n \in I$(countable additivity)

then I is said to be σ-ideal. If $X \notin I$, then I is called a proper ideal and $\{X - E : E \in I)$ is a filter, and hence proper ideals are sometimes called dual filters. We will denote by (X,τ,I) a non empty set X, a topology τ on X, and an ideal I on X. If (X,τ,I) is a space we denote by $\tau^*(I)$ the topology on X generated by the basis $\beta(I,\tau) = \{U - E : U \in \tau, E \in I\}$[34]. When there is no ambiguity we will simply write τ^* for $\tau^*(I)$ and β for $\beta(I,\tau)$ respectively. Examples are provided in [33] and [14] showing that β is not, in general, a topology. The closure operator in τ^*, denoted by Cl^* can be described as follows: For $A \subseteq X$, $(Cl^*(A) = A \cup A^*(I,\tau)$ where $A^*(I,\tau) = \{x \in X : U \cap A \notin I$ for every $U \in N(x)\}$ is called the local function of A with respect to I and τ. We will write A^* for $A^*(I,\tau)$ when no ambiguity is present. In [24], Natkaniec defines an operator $\Psi(I,\tau) : P(X) \to \tau$ where (X,τ,I) is a space as follows: for every $A \subseteq X$, $\Psi(A) = \{x :$ there exists an open nbd U of x such that $U - A \in I\}$.

If $(X < \tau)$ is a space, $A \subseteq X$ and $A \notin \tau$ then the class $\{G \cup (G' \cap A) : G, G' \in \tau\}$ is a topology finer than τ called the simple expansion of τ by A and denoted by $\tau(A)$ [17]. If (X,τ) is a space and $A \subseteq X$, then teh class $\tau[A] = \{U - B : U \in \tau,\ A \supseteq B\}$ is a topology finer than τ called the local discrete expansion of τ by A [35]. In 1970, A. S. Mashhour [19] introduced the lower separation axioms T_0', T_1', T_1'' and T_2'. The definitions of these axioms are based on the basic lower separation axioms and the boundary operator on a set. Recall that a space (X,τ) is said to be extremally disconnected (briefly E. D.) if the closure of every open set of X is open in X. Spaces having only the property that their dense subsets are open are called submaximal. A space (X,τ) is said to be nearly compact [31] (resp. strongly compact [2], semi compact [8], α-compact [22]) if each regular open (resp. preopen, semi-open, α-open) cover has a finite subcover. A space (X,τ) is called P_1 paracompact [21] if every preopen cover of X has a locally finite open refinement.

A bijection $f : X \to Y$ is a semi-homeomorphism if f and f^{-1} preserve semi-open sets [6]. A space (X,τ) is called inverible [9] (resp. semi-invertible [7]) if for each proper open (resp. semi-open) set U in (X,τ) there exists a homeomorphism. (resp. semi homeomorphism) $h : (X,\tau) \to (X,\tau)$ such that $h(X - U) \subset U$.

3. On the *-topology

Theorem 3.1. *Let (X,τ,I) be a space. Then we have:*

(i) $\tau^*(I) = \{U \subseteq X : Cl^*(X - U) = X - U\}$[14].

(ii) $\tau^*(I) = \{A \subseteq X : A \subseteq \Psi(A)\}$[11].

(iii) $\tau^*(I) = \cup\{(G - E) : U \in \tau$ *and* $E \in I\}$.

(iv) $\tau^*(I) = \{U \cap E^c : U \in \tau, E \in I\}$ *where I is a proper ideal on X.*

Corollary 3.1. *For every $A \subseteq (X,\tau,I)$, we have:*

(i) *If $I = \{\Phi\}$, then $A^* = Cl(A)$ and $Cl^*(A) = Cl(A)$ and hence in this* *case* $\tau^*(I) = \tau$.

(ii) *If* $I = P(X)$, *then* $A^* = \Phi$, *and hence* $\tau^*(I)$ *is the discrete topology.*

Corollary 3.2.

(i) *If* I *and* J *are ideals on* (X, τ) *such that* $J \supseteq I$ *then,* $\tau^*(J) \supseteq \tau^*(I)$.

(ii) $\tau^*(I) = \tau$, *if* $E^c \subseteq \tau$ *and* $E \in I$ *where,* E^c *denotes the complement of* E.

(iii) $\tau^*(I) = \tau$ *iff every member of* I *is* τ*-closed* [30].

Remark 3.1. Simple extension $\tau(A)$, local discrete extension $\tau[A]$ and $\tau^*(I)$ are three independent concept as the following example shows.

Example 3.1. Let $X = \{a, b, c, d,\}$ with a topology $\tau = \{X, \Phi, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d,\}\}$, Then for an ideal $I = \{\Phi, \{c\}, \{d\}, \{c, d\}\}$ on X and a subset $A = \{a, b\}$, we can easily deduce that:

(i) $\tau(A) = \{X, \Phi, \{c\}, \{a, c\}, \{c, d\}, \{a, b\}, \{a, c, d\}\{a, b, c\}\}$,

(ii) $\tau[A] = \{X, \Phi, \{c\}, \{a, c\}, \{c, d\}\{a, c, d\}, \{b, c, d\}\}$, and

(iii) $\tau^*(I) = \{X, \Phi, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c\}\}$.

Remark 3.2. Two different topolgies τ_1 and τ_2 on a set X may have the same *-topology $\tau_1^*(I)$ and $\tau_2^*(I)$ where I is the ideal on a nonempty set X. (Example 3.2).

Example 3.2. Let $X = \{a, b, c, d\}$ with two topologies $\tau_1 = \{X, \Phi, \{a\}\}$ and τ_2 be an indiscrete topology. Then for an ideal $I = \{\Phi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$. We notice that $\tau_1^*(I) = \{X, \Phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\} = \tau_2^*(I)$.

Theorem 3.2. *If* (X, τ) *and* $(X, \tau^*(I))$ *are two spaces, then for every* $A \subseteq X$, *we have:*

(i) $d_\tau(A) \supseteq d_{\tau^*(I)}(A)$, *where* d *denotes the derived set of* A.

(ii) $b_\tau(A) \supseteq b_{\tau^*(I)}(A)$, *where* b *denotes the boundary of* A.

Proof.

(i) If $x \notin d_\tau(A)$, then there exists $G \in \tau$ such that $(G - \{x\}) \cap A = \Phi$, but $G \in \tau \subseteq \tau^*(I)$. So $x \notin d_{\tau^*(I)}(A)$.

(ii) Follows directly from the fact that: $Cl_\tau(A) \supseteq Cl_{\tau^*(I)}(A)$, for every $A \subseteq X$.

Remark 3.3. The converse of Theorem 3.2 is not true in general, as shown by the following example.

Example 3.3. Let $X = \{a, b, c\}$ with topology $\tau = \{X, \Phi, \{a\}\}$. Then for an ideal $I = \{\Phi, \{a\}, \{c\}, \{a, c\}\}$ and $A = \{b, c\}$. We notice that: $d_\tau(A) = \{b, c\}$ and $d_{\tau^*(I)}(A) = \{c\}$. Therefore $d_\tau(A) \not\subseteq d_{\tau^*(I)}(A)$.

Theorem 3.3. *If (X, τ) is submaximal, then: $PO\,(X, \tau^*(I)) \supseteq PO\,(X, \tau)$.*

Proof. For a submaximal space, $PO(X, \tau) = \tau$[32]. Then $PO(X, \tau) = \tau \subseteq \tau^*(I) \subseteq PO(X, \tau^*(I))$.

Remark 3.4. Submaximality in Theorem 3.3 is necessary as shown by the following example.

Example 3.4. Let $X = \{a, b, c\}$ with an indiscrete topology τ.

Then, $PO(X, \tau) = P(X)$. For an ideal $I = \{\Phi, \{c\}\}, \tau^*(I) = \{X, \Phi, \{a, b\}\}$, and $PO(X, \tau^*(I)) = \{X, \Phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Therefore, $PO(X, \tau^*(I)) \not\supseteq PO(X, \tau)$.

One can deduce easily the following result which has obvious proof.

Lemma 3.1. *In a space (X, τ), $A \subseteq X$ is preopen iff there exists an open set $G \in \tau$ such that $Cl_\tau(A) \supseteq G \supseteq A$.*

Theorem 3.4. *Let (X, τ) be a space with an ideal I on X and $E^c \in PO(X, \tau)$ for every $E \in I$. Then $PO(X, \tau) \supseteq PO(X, \tau^*(I))$.*

Proof. Let $B \in PO(X, \tau^*(I))$. Then by Lemma 3.1, there exists $G \in \tau^*(I)$ su3ch that $B \subseteq G \subseteq Cl_{\tau}*(B) \subseteq Cl_\tau(B)$, but $G \in \tau^*(I)$ implies $G = U \cap E^C$, $U \in$

τ, $E \in I$, and hence. $B \subseteq U \cap E^C \subseteq Cl_\tau(B)$. Since $E^c \in PO(X,\tau)$, for every $E \in I$, $B \subseteq U \cap E^c \subseteq U \cap \mathrm{Int}_\tau Cl_\tau E^C \subseteq U \cap Cl_\tau\, E^C \subseteq Cl_\tau(U \cap E^C) \subseteq Cl_\tau(B)$. Therefore, $U\cap\ \mathrm{Int}_\tau Cl_\tau E^c$ is a τ-open set containing B and $B \in PO(X,\tau)$.

Combining the previous two theorems, we obtain the following corollary.

Corollary 3.3. *Let (X,τ,I) be a space and $E^c \in PO(X,\tau)$ for every $E \in I$. If X is submaximal, then $PO(X,\tau) = PO((X,\tau^*(I))$.*

Theorem 3.5. *If (X,τ) is E. D., then $SO((X,\tau^*(I)) \supseteq SO\,(X,\tau)$.*

Proof. For an E. D. space $SO\,(X,\tau) = \tau$. Then $SO(X,\tau) = \tau \subseteq \tau^*(I) \subseteq SO\,((X,\tau^*(I))$.

Theorem 3.6. *For a space (X,τ), we have $\alpha O((X,\tau^*(I)) \supseteq \alpha O(X,\tau)$ iff every nowhere dense subset is closed.*

Proof. Follows from the fact that $\tau = \tau^\alpha$ iff every nowhere dense subset is closed.

Theorem 3.7. *Let (X,τ) be a $T_2\,(T_1,\, T_0)$ space, then $(X,\, \tau^*\,(I))$ is $T_2\,(T_1,\, T_0)$ space.*

Proof. Obvious since $\tau \subseteq \tau^*(I)$ and $Cl_{\tau^*}(U) \subseteq Cl_\tau(U)$ for any $U \subseteq X$.

Remark 3.5. The converse of Theorem 3.7 is not true, in general as shown by the following example.

Example 3.5. Let $X = \{a,b,c\}$ with a topology $\tau = \{X,\Phi,\{a\}\}$. Then for an ideal $I = \{\phi,\{b\},\{c\},\{b,c\}\}$ we have $\tau * (I) = \{X,\Phi,\{a\},\{a,b\},\{a,c\}\}$ and we observe that $(X,\tau^*(I))$ is T_0 while (X,τ) is not.

Theorem 3.8. *Let (X,τ) be a $T_2'(T_1',T_1'',T_0')$-space, then $(X,\tau^*(I))$ is $T_2'(T_1',T_1'',T_0')$.*

Proof. We prove the theorem for a T_2'-space. Let (X,τ) be T_2' Then for every $x,y \in X$, $x \neq y$, there exist $U,V \in \tau \subseteq \tau^*(I)$ with $x \in U$ and $y \in V$ such that $b_\tau U \cap b_\tau V = \Phi$. But $b_\tau * U \cap b_\tau * V \subseteq b_\tau U \cap b_\tau V = \Phi$. This implies that $(X, \tau^*(I))$ is T_2'.

Theorem 3.9. *If $(X, \tau^*(I))$ is semi-regular, then:*
(i) (X, τ) *is* $T_2(T_1, T_0)$ *iff* $(X, \tau^*(I))$ *is* $T_2^{(}T_1, T_0)$,
(ii) (X, τ) *is* $T_2'(T_1', T_1'', T_0')$ *iff* $(X, \tau^*(I))$ *is* $T_2'(T_1', T_1'', T_0')$.

Proof. Follows from the fact, if $(X, \tau^*(I))$ is semi-regular then $\tau = \tau * (I)$.

Remark 3.6. The previous theorem is vaild by replacing the condition of semi-regular by the condition, if every member of I is τ-closed.

Remark 3.7. In general, regularity and normality are not preserved under $\tau^*(I)$ as shown by the following examples.

Example 3.6. Let $X = \{a, b, c\}$ with a topology $\tau = \{X, \Phi, \{a\}, \{b, c\}\}$. Then for an ideal $I = \{\Phi, \{b\}\}$, we have $\tau^*(I) = \{X, \Phi, \{c\}, \{c\}, \{a, c\}\{b, c\}\}$. It is clear that (X, τ) is regular while $(X, \tau^*(I))$ is not.

Example 3.7. Let X be as shown in Example 3.6 and $\tau = \{X, \Phi, \{a\}, \{a, b\}\}$. Then for an ideal $I = \{\Phi, \{b\}, \{c\}, \{b, c\}\}$ we notice $\tau^*(I) = \{X, \Phi, \{a\}, \{a, b\}\{a, c\}\}$ and (X, τ) is normal while $(X, \tau^*(I))$ is not.

Theorem 3.10. *If (X, τ, I) is submaximal and $E^c \in PO(X, \tau)$ for every $E \in I$, then (X, τ) is strongly compact iff $(X, \tau^*(I))$ is strongly compact.*

Proof. The first implication is obvious. To prove the second implication, let $\{U_\alpha : \alpha \in \nabla\}$ be a τ^*-preopen cover of $(X, \tau^*(I))$, then $\{U_\alpha : \alpha \in \nabla\}$ is a τ-preopen cover (Corollary 3.3), there exists then a finite subfamily ∇_0 of ∇ such that $X = \cup\{U_\alpha : \alpha \in \nabla_0\}$.

Theorem 3.11. *If $f : (X, \tau^*) \to (Y, \sigma)$ is open, then, $f : (X, \tau, I) \to (Y, \sigma)$ is open.*

Proof. Obvious.

Since the intersection of two regular open sets is regular open, the family of regular open sets forms a base for a smaller topology τ_s on X, called the semi-regularization of τ. The space (X,τ) is said to be semi-regular if $\tau_s = \tau$.

Definition 3.1. [25] Given a space (X,τ,I), I is said to be τ-boundary if $I \cap \tau = \{\Phi\}$.

Theorem 3.12. *Let (X,τ,I) be a space and I is τ-boundary, then $\tau_S = (\tau^*)_S = (\tau_S)_S$.*

Proof. Follows from Theorem 6.4 [14] and Lemma 3 [23].

Theorem 3.13. *If (X,τ,I) is a space such that I is τ-boundary and (X,τ^*) is semiregular, then $\tau^* = (\tau^*)_S$ and $\tau^* = \tau$.*

Proof. Since (X,τ^*) is semiregular, then $\tau^* = (\tau^*)_S$, since I is τ-boundary, then τ_S $(\tau^*)_S$ [14]. Hence $\tau^* = (\tau^*)_S = \tau_S \subseteq \tau \subseteq \tau^*$ and by using Theorem 3.12 we get the result.

Theorem 3.14. *If (X,τ,I) is a space and $I \cap \tau = \{\phi\}$, then $RO(X,\tau) = RO(X,\tau^*)$.*

Proof. The result follows immediately from the fact if (X,τ) is a space and $I \cap \tau = \{\phi\}$, then $\tau_S = (\tau^*)_S$.

Corollary 3.4. *If (X,τ,I) is a space and $X = X^*$, then $RC(X,\tau) = RC(X,\tau^*)$.*

Corollary 3.5. *If (X,τ,I) is a space and I is τ-boundary, then (X,τ) is nearly compact iff (X,τ^*) is nearly compact.*

4. On the toplogy generated by a given topology and ideal of nowhere dense sets

The following Lemma is very useful in the sequel.

Lemma 4.1. [14] *Given a topology τ on X, the α-topology for τ is $\tau^\alpha = \tau^*(I_n)$, where $I_n(\tau)$ is the ideal of nowhere dense sets.*

Theorem 4.1. *If (X,τ) is a space. Then: $\tau = \tau^*(I_n)$ iff every nowhere dense subset is closed.*

Proof. Obvious.

Theorem 4.2. *If (X,τ,I_n) is a space. Then:*

(i) *$\tau^*(I_n)$ is a topology on X and $\tau = \tau^*(I_n)$,*

(ii) *$(\tau^*(I_n))^*(I_n) = \tau^*(I_n)$,*

(iii) *(X,τ) and $(X,\tau^*(I_n))$ have the same class of semi-open sets,*

(iv) *Let σ be any topology on X such that $SO(X,\sigma) = SO(X,\tau)$. Then $\tau^*(I_n) \supseteq \sigma$.*

Proof. Follows from Lemma 4.1 and Theorem 1.2 [4].

Theorem 4.3. *Let (X,τ) be a space. Then $\tau = \tau^*(I_n) = \tau^\alpha = SO(X,\tau)$ if one of the following holds:*

(i) *(X,τ) is a partition topology,*

(ii) *(X,τ) is a cofinite topology,*

(iii) *(X,τ) is the two point Sierpinski space,*

(iv) *$A \in \tau$ or $\text{Int}(A) = \phi$, for every $A \subseteq X$.*

Proof.

(i) Follows from Lemma 4.1 and proposition 3 [28].

(ii) and (iii) Follows from Lemma 4.1 and Corollary 2 [28].

(iv) Obvious from Lemma 4.1 and proposition 2 [28].

Theorem 4.4. *If (X,τ) is submaximal and E. D. Then*

$$\tau = \tau^*(I_n) = \tau^\alpha = SO(X,\tau) = PO(X,\tau) = \beta O(X,\tau).$$

Proof. The result follows from Lemma 4.1 and [3].

Theorem 4.5. *A space (X,τ) is discrete iff $(X,\tau^*(I_n))$ is discrete.*

Proof. If (X,τ) is discrete, then $\tau^*(I_n) \supseteq \tau$. This implies that $(X,\tau^*(I_n))$ is discrete. Conversely see proposition 1 [28].

Theorem. 4.6. *If X is finite and $i \geq 1$, Then (X,τ) is T_i iff $(X,\tau^*(I_n))$ is T_i.*

Proof. If X is finite, then (X,τ) is T_i iff (X,τ) is discrete and hence by Theorem 4.5 iff $(X,\tau^*(I_n))$ is discrete and therefore iff $(X,\tau^*(I_n))$ is T_i.

The following example shows that Theorem 4.6 is false when $i = 0$.

Example 4.1. Let $X = \{a,b,c\}$ and $\tau = \{X,\Phi,\{a\}\}$. Then we have $I_n = \{\Phi,\{b\},\{c\},\{b,c\}\}$ and we notice that $\tau^*(I_n) = \{X,\Phi,\{a\},\{a,b\},\{a,c\}\}$, so that (X,τ) is not T_0, while $(X,\tau^*(I_n))$ is T_0.

Theorem 4.7. *(X,τ) is a Hausdorff space iff $(X,\tau^*(I_n))$ is Hausdorff.*

Proof. One implication is immediate, since $\tau^*(I_n) \supseteq \tau$. Conversely, follows from Lemma 4.1 and Theorem 2 [28].

Theorem 4.8. *If A is a subset of (X,τ,I_n) then:*

(i) $Cl_\tau(A) \supseteq Cl_{\tau^*}(A) \supseteq A^*(I_n,\tau)$,

(ii) *$A \in \tau^*(I_n)$ iff there exists a regular open set G in (X,τ) and a nowhere dense set N such that $A = G - N$.*

Proof. (i) Obvious from Lemma 4.1 and Proposition (1) [29]. (ii) The result follows from Lemma 4.1 and Proposition (4) [26].

Theorem 4.9. *If (X,τ) is P_1-paracompact. Then:*

(i) *$\tau^*(I_n) = \tau^*(I_f) = PO(X,\tau)$, when X is T_i,*

(ii) *$\tau^*(I_n) = \tau^*(I) = PO(X,\tau)$ iff every member of I is closed in (X,τ).*

Proof.

(i) Follows from Lemma 4.1, see [14] and the fact that $\tau^*(I_f) = \tau$ if (X,τ)

is T_1.

(ii) Obvious from [14] and [10].

Theorem 4.10. *If (X,τ) is a T_1-space. Then:*

(i) $\tau^*(I_n) = \tau^*(I_f)$ *iff every nowhere dense subset is closed,*

(ii) $PO(X,\tau) \supseteq \tau^*(I_n) \supseteq \tau^*(I_f)$.

Proof.

(i) Obvious.

(ii) Follows from [5] and the fact that $\tau^*(I_f) = \tau$ if (X,τ) is T_1.

Theorem 4.11. *Let (X,τ,I_n) be a space, then the following properties are satisfied,*

(i) $SO(X,\tau) = SO(X,\tau^*(I_n))$,

(ii) $PO(X,\tau) = PO(X,\tau^*(I_n))$,

(iii) $RO(X,\tau) = RO(X,\tau^*(I_n))$,

(iv) (X,τ) *and* $(X,\tau^*(I_n))$ *have the same class of nowhere dense sets.*

(v) (X,τ) *and* $(X,\tau^*(I_n))$ *have the same class of dense sets.*

Proof. It is obvious by using Lemma 4.1 and Theorem 2.9 [4].

Corollary 4.1. *Let (X,τ,I_n) be a space, then the following properties are satisfied:*

(i) $SC(X,\tau) = SC(X,\tau^*(I_n))$,

(ii) $PC(X,\tau) = PC(X,\tau^*(I_n))$, *and*

(iii) $RC(X,\tau) = RC(X,\tau^*(I_n))$.

Corollary 4.2. *For a space (X,τ) we have:*

(i) (X,τ) *is semi-compact iff* $(X,\tau^*(I_n))$ *is semi-compact,*

(ii) (X,τ) *is strongly-compact iff* $(X,\tau^*(I_n))$ *is strongly-compact,*

(iii) (X,τ) *is nearly-compact iff* $(X,\tau^*(I_n))$ *is nearly-compact.*

Proof. Follows from Theorem 4.11.

Theorem 4.12. *A space $(X, \tau^*(I_n))$ is submaximal iff each codense set in (X, τ) is nowhere dense in (X, τ).*

Proof. Let A be a codense set in (X, τ). Then A is codense in $(X, \tau^*(I_n))$ and Since (X, τ^*) is submaximal, then A is closed in $(X, \tau^*(I_n))$, thus A is nowhere dense in (X, τ^*) and hence A is nowhere dense in (X, τ).

Conversly, let D be a dense set in $(X, \tau^*(I_n))$ Then D is dense in (X, τ). So that X-D is codense in (X, τ). By hypothesis, X-D is nowhere dense in (X, τ) and hence is nowhere dense in $(X, \tau^*(I_n))$. Since nowhere dense sets in $(X, \tau^*(I_n))$ are closed in $(X, \tau^*(I_n))$. D is open in $(X, \tau^*(I_n))$. Then $(X, \tau^*(I_n))$ is submaximal.

Theorem 4.13. *A space (X, τ) is semi-invertible iff $(X, \tau^*(I_n))$ is invertible.*

Proof. Follows From Lemma 4.1 Theorem 3.4 [18].

Theorem 4.14. *A topological property is semitopological iff it is shared by a space (X, τ) and the space $(X, \tau^*(I_n)$.*

Proof. Obvious from Lemma 4.1 and see [13].

Theorem 4.15. *If (X, τ) is a semi-invertible space and contains a non-empty set $U \in \tau^*(I_n)$ which is Urysohn (resp. E. D.) as a subspace of (X, τ), then (X, τ) is Urysohn (resp. E. D.)*

Proof. Follows From Lemma 4.1 and Corollary 3.6 in [18].

The following result is obvious from Lemma 4.1 and the definition of α-compact space.

Theorem 4.16. *A space (X, τ) is α-compact iff $(X, \tau^*(I_n))$ is compact.*

Corollary 4.3. *A subset A of a space (X, τ) is α-compact relative to (X, τ) iff it is compact in $(X, \tau^*(I_n))$.*

Corollary 4.4. *A function $f:(X,\tau)\to(Y,\sigma)$ is a semi-homeomorphism iff $f:(X,\tau^*(I_n))\to(Y,\sigma^*(I_n))$ is a homeomorphism.*

Proof. Follows from the fact that semi-topological properties and α-topological properties are conincide [6].

Here we prove that the property of connectedness is shared by any topological space and its *-topology.

Theorem 4.17. *If (X,τ) is a topological space, then (X,τ) is disconnected iff $(X,\tau^*(I_n))$ is disconnected.*

Proof. If (X,τ) is disconnected, then $\tau^*(I_n)\supseteq\tau$. This implies that $(X,\tau^*(I_n))$ is disconnected.

Conversely follows immediately from Lemma 4.1 and Theorem 2 in [27].

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