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COMMUTATIVITY OF RIGHT S-UNITAL RINGS UNDER SOME POLYNOMIAL CONSTRAINTS

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Abstract. In the present paper we discuss the commutativity of certain rings, namely rings with unity 1 and right s-unital rings under each of the following conditions: $(P_1)[yx^m - x^n f(y), x] = 0$, $(P_1)^*[yx^m - f(y)x^n, x] = 0$, where m, n are fixed non-negative integers and $f(\mathrm{x})$ is a polynomial in $X^2\mathbb{Z}(X)$ varying with the pair of ring elements x, y. Further, the results have been extended to the case when m and n depend on the choice of x and y and the ring satisfies the Chacron's condition.

1. Introduction

Following [7], an associative ring R is said to be right (resp. left) s-unital, if for each element x in R, $x \in xR$ (resp. $x \in Rx$); R is called s-unital if $x \in xR \cap Rx$. There are enough examples (to mention a few [1, Examples 1-2] and [10, Remark 2]) which show that these classes of rings are generalizations of the class of rings with unity 1. Recently many results for rings with unity 1 particularly a number of commutativity theorems have been extended to one sided s-unital rings.

We remark, incidentally that as usual for any pair of ring elements x and y, $[x, y] = xy - yx$. In Section 2, we investigate the commutativity of rings with unity 1 under the ring properties:

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(P_1) For every x, y in R there exists $f(X) \in X^2\mathbb{Z}[X]$, such that $[yx^m - x^n f(y), x] = 0$, where m, n are fixed non-negative integers.

$(P_1)^*$ For every x, y in R there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $[yx^m - f(y)x^n, x] = 0$, where m, n are fixed non-negative integers.

(P_2) For every x, y in R there exist non-negative integers m, n and $f(X) \in X^2\mathbb{Z}(X)$ such that $[yx^m - x^n f(y), x] = 0$.

$(P_2)^*$ For every x, y in R there exist non-negative integers m, n and $f(X) \in X^2\mathbb{Z}[X]$ such that $[yx^m - f(y)x^n, x] = 0$.

(CH) For every x, y in R there exist $f(X), g(X)$ in $X^2\mathbb{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$.

The results obtained are further extended to the right s-unital rings in the subsequent section.

2. Commutativity of Rings with Unity 1

Consider the following types of rings:

(a) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

$(a)_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(b) $M_\sigma(K) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} / a, b \in K \right\}$, where K is a finite field with a non-trivial automorphism σ.

(c) A non-commutative division ring.

(d) $S = \langle 1 \rangle + T$, T is a non-commutative radical subring of S.

(e) $S = \langle 1 \rangle + T$, T a non-commutative subring of S such that $T[T, T] = [T, T]T = 0$.

Very recently Streb [9] gave a classification for non-commutative rings, which provide a powerful tool in obtaining a number of commutativity theorems (cf. [4], [5], [6] and [8]). One can easily observe from the proof of [9, Corollar (1)] that if R is a non-commutative ring with unity 1, then there exists a factor subring of R which is of type (a), (b), (c), (d) or (e). Thus we have the following:

Lemma 1. *Let P be a ring property, which is inherited by factor subrings. If no rings of type* (a), (b), (c), (d) *or* (e) *satisfy P, then every ring with unity 1 satisfying P is commutative.*

We apply the above lemma to prove the following theorems:

Theorem 1. *If R is a ring with unity* 1 *satisfying any one of the properties* (P_1) *and* $(P_1)^*$, *then R is commutative (and conversely).*

Theorem 2. *Let R be a ring with unity* 1 *satisfying* (CH). *Suppose further that R satisfies any one of the properties* (P_2) *and* $(P_2)^*$. *Then R is commutative (and conversely).*

In preparation for the proofs of the above theorems, we begin with the following lemma due to Herstein [2]. Perhaps it is sufficient to prove Theorem 1 for the rings satisfying the property (P_1) and Theorem 2 for the property (P_2). The proofs for the cases $(P_1)^*$ and $(P_2)^*$ will follow on the same lines.

Lemma 2. (Herstein [2]). *If for every x, y in R there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $[x - f(x), y] = 0$, then R is commutative.*

Proof of Theorem 1. Suppose that R satisfies the property (P_1). First, we consider the rings of type (a). Then in $M_2(GF(p))$, p a prime, we find that $[e_{12}e_{22}^m - e_{22}^n f(e_{12}), e_{22}] \neq 0$ for all integers $m \geq 0$, $n \geq 0$ and $f(\mathrm{x}) \in X^2\mathbb{Z}[X]$.

Next, consider the ring $M_\sigma(K)$. Let

$$x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} \ (a \neq \sigma(a)) \quad \text{and} \quad y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $[yx^m - x^n f(y), x] = -[x, y]x^m = (\sigma(a) - a)\ (\sigma(a))^m y \neq 0$ for all integers $m \geq 0$, $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$.

Further, let R be a ring of type (c). If x is a unit of R, then for every y in R choose $f(X) \in X^2\mathbb{Z}[X]$ such that $[yx^{-m} - x^{-n}f(y), x^{-1}] = 0$ i.e. $[yx^{-m} - x^{-n}f(y), x] = 0$. This implies that

$$x^n[x, y] = [x, f(y)]x^m \tag{1}$$

Next, choose $p(X) \in X^2\mathbb{Z}[X]$ such that $[f(y)x^m - x^n p(f(y)), x] = 0$. Hence, we get

$$[x, f(y)]x^m = x^n[x, p(f(y))] \tag{2}$$

Compare (1) and (2), to get $x^n[x,y] = x^n[x,h(y)]$, where $h(X) = p(f(X)) \in X^2\mathbb{Z}[X]$. Since, x is a unit $[x, y - h(y)] = 0$ and by Lemma 2, R is commutative, a contradiction.

Suppose that R has a factor subring of type (d). Let $s, t \in T$. Since $1 - s$ is a unit, by above paragraph there exists $f(X) \in X^2\mathbb{Z}[X]$ such that

$$[s, t - f(t)] = -[1 - s, t - f(t)] = 0.$$

Hence, T is commutative by Lemma 2, again a contradiction.

Finally, consider $S = <1> + T$, where T is a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$. Suppose that R has a factor subring S. Now choose $s, t \in T$ such that $[s,t] \neq 0$. Then there exists $f(X) \in X^2\mathbb{Z}[X]$ such that

$$[s,t] = [s,t](1+s)^m = (1+s)^n[s, f(t)] = 0,$$

a contradiction.

Thus we have seen that no rings of type (a), (b), (c), (d) or (e) satisfy (P_1). Hence, by Lemma 1, R must be commutative.

Proof of Theorem 2. As above, it is easy to see that no rings of type (a) or (b) satisfy (P_2). Combining this fact with the Corollary 1 of [4], we get the required result.

3. Commutativity of Right s-Unital Rings

Dualizing the proof of [6, Lemma 1], we can prove the following lemma.

Lemma 3. *If R is right s-unital and not left s-unital, then R has a factor subring of type $(a)_r$.*

If R is a right s-unital ring satisfying (P_1), then a careful scrutiny of the first paragraph of the proof of Theorem 1 shows that no rings of type $(a)_r$ satisfy (P_1). Hence, by Lemma 3, R is s-unital and in view of Proposition 1 of [3], we may assume that R has unity 1. Combining this fact together with Theorem 1, we get the following:

Theorem 3. *If R is a right s-unital ring satisfying any one of the properties (P_1) and $(P_1)^*$, then R is commutative (and conversely).*

Corollary 1. *Let $m \geq 0$, $n \geq 0$ be fixed integers. If R is a right s-unital ring in which for every x, y in R there exists integer $q = q(x,y) > 1$ such that either $[yx^m - x^n y^q, x] = 0$ or $[yx^m - y^q x^n, x] = 0$, then R is commutative (and conversely).*

Similarly, by using [4, Corollary 1], we can prove the following:

Theorem 4. *Let R be a right s-unital ring satisfying* (CH) *Suppose further that R satisfies any one of the properties (P_2) and $(P_2)^*$. Then R is commutative (and conversely).*

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