TAMKANG JOURNAL OF MATHEMATICS
Volume 24, Number 1, Spring 1993

ON COMMUTATIVITY THEOREMS FOR P. I. – RINGS WITH UNITY

THOMAS P. KEZLAN

Abstract. The purpose of this paper is to show how a previous commutativity theorem for general rings can be used to prove commutativity theorems for rings with unity, and to obtain several new results via this route, e.g., if a ring with unity satisfies either $x^k[x^n,y]=[x,y^m]x^\ell$ or $x^k[x^n,y]=[x,y^m]y^\ell$ $(m>1)$ and if either (A) m and n are relatively prime or (B) $n[x,y]=0$ implies $[x,y]=0$, then R is commutative.

There are numerous results in the literature [1, 7, 9, 10, 11, 12] concerning the commutativity of rings satisfying special cases of the identities

$$x^k[x^n,y] = [x,y^m]x^\ell, \quad \text{or} \tag{$*$}$$

$$x^k[x^n,y] = [x,y^m]y^\ell, \tag{$**$}$$

where $m>1$.

In this paper we offer a simultaneous generalization of these results for rings with unity as well as several other results which further illustrate the method used.

Theorem 1. *Let R be a ring with unity satisfying either $(*)$ or $(**)$ where k and ℓ are non-negative integers and m and n are positive integers such that*

Received October 15, 1991.
AMS (MOS) Subject Classification: 16A70, 16A38.

either

$$m \text{ and } n \text{ are relatively prime, or} \tag{A}$$

$$n[x,y]=0 \text{ implies } [x,y]=0 \text{ in } R. \tag{B}$$

Then R must be commutative.

The main tool used in this paper is the result in [6], the statement of which we repeat here as

Lemma 1. ([6]) *If R is a ring satisfying an identity of the form*

$$[x,y]+F(x,y) \;=\; 0$$

where each homogeneous component of F has (integer) coefficients totaling 0 and where F either has no linear terms in x or has none in y, then R is commutative.

An immediate corollary of Lemma 1 is

Lemma 2. *Let R be a ring, p a prime such that $p[x,y]=0$ for all x,y in R, and n a positive integer not divisible by p, and suppose that R satisfies an identity of the form*

$$n[x,y]+F(x,y) \;=\; 0$$

where F satisfies the same conditions as in Lemma 1. Then R must be commutative.

Proof. If $sp+tn=1$, then multiply the identity by t and use $p[x,y]=0$ to obtain

$$[x,y]+tF(x,y) \;=\; 0,$$

whence Lemma 1 applies.

The next two lemmas are well-known and their proofs will be omitted.

Lemma 3. [5, p. 221] *If $[[x,y],y]=0$, then*

$$[x,y^n] \;=\; n[x,y]y^{n-1}$$

for all positive integers n.

Lemma 4. [8] *If $f : R \to R$ satisfies $f(x+1) = f(x)$ and $f(x)x^{\ell} = 0$ for all $x \in R$, then $f(x) = 0$ for all $x \in R$.*

Proof of Theorem 1. By the result in [3] any ring satisfying either (*) or (**) has nil commutator ideal $C(R)$: Take $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We divide the rest of the proof into two cases.

Case I: R satisfies (*). By way of contradiction suppose there exists a non-commutative ring with unity satisfying (*). The next step is to pass to the subdirectly irreducible case, and under hypothesis (A) this reduction can be achieved as in [6]; that is, without loss of generality we assume there is a ring R such that

(1) R is non-commutative with unity, satisfies (*), and is subdirectly irreducible with heart $H = C(R)$ and with $H^2 = (0)$.

Under hypothesis (B) however, we must perform a slight variation on the argument in [6] since this hypothesis is not preserved by homomorphism. Assuming then that we have a non-commutative ring S with unity satisfying (*) and that (B) holds, we proceed as follows: Choose a, b, in S with $n[a, b] \neq 0$ and use Zorn's Lemma to obtain an ideal M which is maximal with respect to the exclusion of $n[a, b]$. Then the ring $\overline{S} = S/M$ is not commutative, satisfies (*), and is subdirectly irreducible with heart containing $n[\overline{a}, \overline{b}]$. Thus, although the ring $R = \overline{S}$ may not inherit condition (B), we can say that n does not annihilate all commutators of R.

Summarizing, we see that if hypothesis (A) holds, then we have a ring R satisfying (1), whereas if (B) holds, then in addition to (1) R satisfies

(2) n does not annihilate all commutators of R.

For fixed y, z in R define $f : R \to R$ by

$$f(x) = [x, (y+z)^m - y^m - z^m] \text{ for all } x \in R.$$

Clearly $f(x+1) = f(x)$, and also

$$\begin{aligned} f(x)x^\ell &= [x,(y+z)^m]x^\ell - [x,y^m]x^\ell - [x,z^m]x^\ell \\ &= x^k[x^n, y+z] - x^k[x^n, y] - x^k[x^n, z] = 0 \end{aligned}$$

whence by Lemma 4 $f(x) = 0$ for all $x \in R$.

Thus

$$[x,(y+z)^m] = [x,y^m] + [x,z^m] \text{ for all } x,y,z \text{ in } R \tag{3}$$

Taking $z = 1$ in (3), we obtain the identity

$$m[x,y] + \sum_{i=2}^{m-1} \binom{m}{i} [x, y^i] = 0. \tag{4}$$

If the additive group of R is torsion-free, then we get commutativity from (4) since the homogeneous components must vanish on R. Thus, again as in [6], the subdirect irreducibility yields a unique prime p such that R has elements of additive order p, from which it follows that $pH = (0)$ (in particular p annihilates all commutators).

If p does not divide m, then (4) provides an identity of the type in Lemma 2, whence R is commutative. Hence p must divide m. Moreover, p cannot divide n, which is obvious if (A) holds, and is also clear if (B) holds since n does not annihilate all commutators whereas p does. Thus in either case (A) or (B) we have

$$p \text{ divides } m \text{ but does not divide } n. \tag{5}$$

For consistency of notation in what follows, we re-write (3), interchanging x and y:

$$[y,(x+z)^m] = [y,x^m] + [y,z^m] \text{ for all } x,y,z \text{ in } R. \tag{6}$$

In (*) we replace x by $x+1$ to obtain the identity

$$\sum_{i=0}^{k} \sum_{j=1}^{n} \binom{k}{i} \binom{n}{j} x^i [x^j, y] = \sum_{i=0}^{\ell} \binom{\ell}{i} [x, y^m] x^i. \tag{7}$$

We next show that $H \subseteq Z$, the center of R. If $x \in H$ and $y \in R$, then recalling that $H^2 = (0)$, we reduce (7) to

$$n[x,y] = [x,y^m] \text{ for } x \in H,\ y \in R, \tag{8}$$

while taking $z = y$ in (6), we obtain

$$[y, xy^{m-1} + yxy^{m-2} + \cdots + y^{m-1}x] = 0, \text{ and hence}$$

$$[x,y^m] = 0 \text{ for } x \in H,\ y \in R. \tag{9}$$

Now (8), (9), and the fact that p does not divide n imply that $[x,y] = 0$ for $x \in H$, $y \in R$, that is

$$H \subseteq Z. \tag{10}$$

Since *all* commutators are now central, we obtain from Lemma 3

$$[x,y^m] = m[x,y]y^{m-1} = 0 \text{ for } all\ x,y \text{ in } R. \tag{11}$$

using the fact that m, being divisible by p, annihilates all commutators.

Hence (7) reduces to

$$n[x,y] + \sum_{j=2}^{n} \binom{n}{j} [x^j, y] + \sum_{i=1}^{k} \sum_{j=1}^{n} \binom{k}{i} \binom{n}{j} x^i [x^j, y] = 0,$$

an identity of the form to which Lemma 2 applies, whence R is commutative.

Case II: R satisfies (**). The reduction to a subdirectly irreducible ring R satisfying (1) in case (A) holds and both (1) and (2) in case (B) holds is valid, as in Case I.

Replacing x by $x+1$ and y by $y+1$ successively in (**) yields the two identities

$$n[x,y] + F(x,y) = 0, \tag{12}$$

$$m[x,y] + G(x,y) = 0. \tag{13}$$

where F and G satisfy the conditions of Lemma 1. Just as in Case I we have a unique prime p with $pH = 0$, so in view of Lemma 2, p must divide both m and n, an impossiblity if (A) holds. On the other hand since n is divisible by p, it must annihilate all commutators, contradicting (2) if (B) holds. This completes the proof.

We suggest that Lemmas 1 and 2 can be used to prove other commutativity theorems for P. I.-rings with unity, and we illustrate this by re-proving several recent results. Consider the

Theorem. (Ashraf and Quadri [2]). *Let R be a ring with unity in which $[xy - x^n y^m, x] = 0$ for all x, y in R and fixed integers $m > 1$, $n \geq 1$. Then R is commutative.*

Proof. Replace x by $x+1$ in the identity to obtain $[x, y] = \sum_{i=0}^{n-1} \binom{n}{i} x^i [x, y^m]$ and use Lemma 1.

The identity in this theorem can be written $x[x, y] = x^n [x, y^m]$, which is a special case of

$$x^k [x^n, y] = x^\ell [x, y^m] \; (m > 1), \qquad (***)$$

and the latter identity, assuming of course the additional hypothesis of condition (A) or (B), can be handled in virtually the same way as (*). Indeed the same applies to

$$x^k [x^n, y] = y^\ell [x, y^m] \; (m > 1), \qquad (****)$$

and so we have, with proof omitted,

Theorem 2. *If R is a ring with unity satisfying $(***)$ or $(****)$ and if either (A) or (B) of Theorem 1 holds, then R is commutative.*

For a final example we re-prove the

Theorem. (Bell, Quadri, and Khan [4]). *Let R be a ring with unity satisfying an identity of the form*

$$[xy - p(xy), x] = 0$$

where $p(X) \in X^2\mathbb{Z}(X)$. Then R is commutative.

Proof. Let $p(X) = \alpha_2 X^2 + \alpha_3 X^3 + \ldots + \alpha_n X^n$ where the α_i are integers. Replacing x by $x+1$ in

$$x[x,y] = [x, p(xy)] \tag{14}$$

yields

$$\begin{aligned}
&(x+1)[x,y] \\
&=[x, \alpha_2((x+1)y)^2 + \alpha_3((x+1)y)^3 + \ldots + \alpha_n((x+1)y)^n] \\
&= (x+1)[x, \alpha_2 y(x+1)y + \alpha_3 y(x+1)y(x+1)y + \ldots \\
&\quad + \alpha_n y(x+1)y(x+1)\ldots(x+1)y] \\
&= x[x, \alpha_2 yxy + \alpha_3 yxyxy + \ldots + \alpha_n yxyx\ldots xy] \\
&\quad + [x, \alpha_2 y^2 + \alpha_3 y^3 + \ldots + \alpha_n y^n] + G(x,y),
\end{aligned}$$

where each homogeneous component of G has coefficients totaling 0, G has no terms linear in y, and each term of G has degree greater than 1 in x. Thus we have

$$x[x,y] + [x,y] = [x, p(xy)] + [x, p(y)] + G(x,y), \tag{15}$$

and subtracting (14) from (15) yields

$$[x,y] = [x, p(y)] + G(x,y),$$

whence R is commutative by Lemma 1.

A similar use of Lemma 1 yields commutativity when R satisfies $[xy - p(yx), x] = 0$.

References

[1] H. A. S. Abujabal, "A generalization of some commutativity theorems for rings I", *Tamkang Journ. Math.*, 21 (1990), 239-245.

[2] Mohd. Ashraf and Murtaza A. Quadri, "On commutativity of associative rings", *Bull. Austral. Math. Soc.*, 38 (1988), 267-271.

[3] Howard E. Bell, "On some commutativity theorems of Herstein", *Arch. Math.*, 24 (1973), 34-38.

[4] H. E. Bell, M. A. Quadri, and M. A. Khan, "Radovi Matematički", 3(1987), 255-260.

[5] Nathan Jacobson, "Structure of Rings", *Amer. Math. Soc. Colloquium Publ.*, 37, Providence, 1964.

[6] Thomas P. Kezlan, "On identities which are equivalent with commutativity", *Math. Japonica*, 29 (1984), 135-139.

[7] Hiroaki Komatsu, "A commutativity theorem for rings", *Math. J. Okayama Univ.*, 26 (1984), 109-111.

[8] W. K. Nicholson and Adil Yaqub, "A commutativity theorem for rings and groups", *Canad. Math. Bull.*, 22 (1979), 419-423.

[9] Evagelos Psomopoulos, "Commutativity theorems for rings and groups with constraints on commutators", *Internat. J. Math. & Math. Sci.*, 7 (1984), 513-517.

[10] Evagelos Psomopoulos, "A commutativity theorem for rings and groups with constraints on commutators", *Glasnik Matematički*, 20 (1985), 7-14.

[11] Evagelos Psomopoulos, "A commutativity theorem for rings with polynomial identities", *Bull. Cal. Math. Soc.*, 77 (1985), 113-114.

[12] Murtaza A. Quadri and Moharram A. Khan, "A commutativity theorem for left s-unital rings", *Bull. Inst. Math. Acad. Sinica*, 15 (1987), 323-327.

University of Missouri-Kansas City, Department of Mathematics, 5100 Rockhill Road, Kansas City, Missouri 64110-2499, U. S. A.