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ON NONLINEAR INTEGRAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. We prove an existence theorem for a class of nonlinear integral equations with deviating arguments.

1. Introduction

Several authors have studied the nonlinear Volterra integral equation with deviating arguments [1, 4, 6, 7, 8]. Banas [5] has proved an existence theorem for functional integral equation. Balachandran [1] has extended this theorem to a class of nonlinear Volterra integral equations with deviating arguments. Balachandran and Ilamaran [2, 3] established existence theorems for nonlinear integral equations with deviating arguments. In this paper we shall derive a set of sufficient conditions for the existence of a solution of a class of nonlinear integral equations with deviating arguments. The technique used in this paper is similar to the one used by Banas [5] and Balachandran and Ilamaran [2, 3].

2. Basic Assumptions

Let $p(t)$ be a given continuous function defined on the interval $[0,\infty)$ and taking real positive values. Denote $C([0,\infty),p(t) : R^n)$ by C_p, the set of all

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continuous functions from $[0, \infty)$ into R^n such that

$$\sup\{|x(t)|p(t) : t \geq 0\} < \infty.$$

It has been proved [8] that C_p forms a real Banach space with regard to the norm

$$||x|| = \sup\{|x(t)|p(t) : t \geq 0\}.$$

If $x \in C_p$ then we will denote $\mathcal{W}^T(x, \varepsilon)$ the usual modulus of continuity of x on the interval $[0, T]$ i.e.,

$$\mathcal{W}^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : |t - s| \leq \varepsilon, t, s \in [0, T]\}$$

Our existence theorem is based on the following lemma.

Lemma. [6] *Let E be a bounded set in the space C_p. If all functions belonging to E are equicontinuous on each interval $[0, T]$ and $\lim_{T \to \infty} \sup\{|x(t)|p(t) : t \geq T\} = 0$ uniformly with respect to E, then E is relatively compact in C_p.*

Consider the nonlinear Volterra integral equation of the form

$$x(t) = H(t, x(t)) + g(t, \int_0^t K(t, s, x(h(s)))ds) \tag{1}$$

where x, H, K and g are n-vectors.

Assume the following conditions:

(i) Let $\Delta = \{(t, s) : 0 \leq s \leq t < \infty\}$

The function $K : \Delta \times R^n \to R^n$ is continuous and there exists continuous functions $m : \Delta \to [0, \infty)$, $a : [0, \infty) \to (0, \infty)$ $b : [0, \infty) \to [0, \infty)$ such that

$$|K(t, s, x)| \leq m(t, s) + a(t)b(s)|x|$$

for all $(t, s) \in \Delta$ and $x \in R^n$.

In order to formulate other assumptions let us define

$$L(t) = \int_0^t a(s)b(s)ds, \quad t \geq 0.$$

Take an arbitrary number $M > 0$ and consider the space C_p with $p(t) = [a(t)e^{ML(t)+t}]^{-1}$.

(ii) there exists a constant $B > 0$ such that for any $t \in [0, \infty)$ the following inequality holds

$$\int_0^t m(t,s)ds \leq Ba(t)e^{ML(t)}$$

(iii) $H : [0, \infty) \times R^n \to R^n$ is continuous and there exists a constant A such that

$$|H(t, x(t))| \leq A|x(t)|$$

(iv) the function $g : [0, \infty) \times R^n \to R^n$ is continuous and satisfies the Lipschitz condition

$$|g(t,x) - g(t,y)| \leq k|x - y|$$

where k is a constant and

$$|g(t,0)| \leq Ra(t)e^{ML(t)}$$

(v) $h : [0, \infty) \to [0, \infty)$ is a continuous function satisfying the condition $L(h(t)) - L(t) \leq N$ where N is a positive constant.

(vi) $a(h(t))/a(t) \leq (M/k)(1 - A - kB - R)e^{-MN}$ where $A + kB + R < 1$

3. Existence Theorem

Theorem. *Assume that the hypotheses* (i) *to* (vi) *hold; then the equation* (1) *has atleast one solution x in the space C_p such that $|x(t)| \leq a(t)e^{ML(t)}$ for any $t \geq 0$.*

Proof. Define a transformation F in the space C_p by

$$(Fx)(t) = H(t, x(t)) + g(t, \int_0^t K(t, s, x(h(s)))ds) \tag{2}$$

From our assumptions we observe that $(Fx)(t)$ is continuous on the interval $[0, \infty)$. Define the set E in C_p by

$$E = \{x \in C_p : |x(t)| \leq a(t)e^{ML(t)}\}.$$

Clearly E is nonempty, bounded, convex and closed in C_p. Now we prove that F maps the set E into itself. Take $x \in E$. Then from our assumptions we have

$$\begin{aligned}
&|(Fx)(t)| \\
&\le |H(t,x(t))| + k\int_0^t |K(t,s,x(h(s)))|ds + |g(t,0)| \\
&\le A|x(t)| + k\int_0^t m(t,s)ds + ka(t)\int_0^t b(s)|x(h(s))|ds + |g(t,0)| \\
&\le Aa(t)e^{ML(t)} + kBa(t)e^{ML(t)} \\
&\quad + ka(t)\int_0^t b(s)a(h(s))e^{ML(h(s))}ds + Ra(t)e^{ML(t))} \\
&\le (A+kB+R)a(t)e^{ML(t)} \\
&\quad + M(1-A-kB-R)a(t)\int_0^t a(s)b(s)e^{ML(s)}e^{-MN}e^{M[L(h(s))-L(s)]}ds \\
&\le (A+kB+R)a(t)e^{ML(t)} + (1-A-kB-R)a(t)\int_0^t Ma(s)b(s)e^{ML(s)}ds \\
&\le (A+kB+R)a(t)e^{ML(t)} + (1-A-kB-R)a(t)e^{ML(t)} \\
&= a(t)e^{ML(t)}
\end{aligned}$$

which proves that $FE \subset E$.

Now we want to prove that F is continuous on the set E. For this let us fix $\varepsilon > 0$ and take $x, y \in E$ such that $\|x-y\| \le \varepsilon$. Further take an arbitrary fixed $T > 0$. In view of (i) and (iv) the functions $K(t,s,x)$ and $H(t,x)$ are uniformly continuous on

$$[0,T]\times[0,T]\times[-r(h(t)),r(h(T))]^n \text{ and } [0,T]\times[-r(T),r(T)]^n$$

respectively, where $r(T) = max\{a(s)e^{ML(s)} : s \in [0,t]\}$. Thus, we have for $t \in [0,T]$

$$\begin{aligned}
|(Fx)(t)-(Fy)(t)| \le& |H(t,x(t)) - H(t,y(t))| \\
&+ k\int_0^t |K(t,s,x(h(s))) - K(t,s,y(h(s)))|ds \\
\le& \beta_1(\varepsilon) + \beta_2(\varepsilon) \qquad (3)
\end{aligned}$$

where β_i are continuous functions such that $\lim_{\varepsilon\to 0}\beta_i(\varepsilon) = 0$. Further, let us take $t \geq T$. Then we have

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq |(Fx)(t)| + |(Fy)(t)| \\
&\leq 2a(t)e^{ML(t)} \\
|(Fx)(t) - (Fy)(t)|p(t) &\leq 2e^{-t}
\end{aligned}$$

Hence for sufficiently large T we have

$$|(Fx)(t) - (Fy)(t)|p(t) \leq \varepsilon \tag{4}$$

By (3) and (4) we get F is continuous on the set E. Hence F is continuous on E.

Now we prove that FE is relatively compact. For every $x \in E$ we have $Fx \in E$ which gives $|(Fx)(t)|p(t) \leq e^{-t}$. Hence $\lim_{T\to\infty} \sup\{|(Fx)(t)|p(t) : t \geq T\} = 0$ uniformly with respect to $x \in E$.

Furthermore, let us fix $\varepsilon > 0$, $T > 0$; $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$. Then for $x \in E$, we have

$$\begin{aligned}
&|(Fx)(t) - (Fx)(s)| \\
&\leq |H(t, x(t)) - H(s, x(s))| \\
&\quad + |g(t, \int_0^t K(t, u, x(h(u)))du) - g(t, \int_0^s K(s, u, x(h(u)))du)| \\
&\quad + |g(t, \int_0^s K(s, u, x(h(u)))du) - g(s, \int_0^s K(s, u, x(h(u)))du)| \\
&\leq \mathcal{W}^T(H, \varepsilon) + k|\int_0^t K(t, u, x(h(u)))du - \int_0^s K(s, u, x(h(u)))du|
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{W}^T(g,\varepsilon)\\
&\leq \mathcal{W}^T(H,\varepsilon)+k\Big|\int_0^t K(t,u,x(h(u)))du-\int_0^s K(t,u,x(h(u)))du\Big|\\
&\quad +k\Big|\int_0^s K(t,u,x(h(u)))du-\int_0^s K(s,u,x(h(u)))du\Big|+\mathcal{W}^T(g,\varepsilon)\\
&\leq \mathcal{W}^T(H,\varepsilon)+k\int_s^t |K(t,u,x(h(u)))|du\\
&\quad +k\int_0^s |K(t,u,x(h(u)))-K(s,u,x(h(u)))|du+\mathcal{W}^T(g,\varepsilon)\\
&\leq \mathcal{W}^T(H,\varepsilon)+k\in\max\{m(t,u)+a(t)b(u)[p(h(u))]^{-1}:0\leq u\leq t\leq T\}\\
&\quad +kT\mathcal{W}^T(K,\varepsilon)+\mathcal{W}^T(g,\varepsilon)
\end{aligned}
$$

which tends to zero as $\varepsilon \to 0$. Thus FE is equicontinuous on $[0,T]$.

Therefore from the lemma FE is relatively compact. Thus the Schauder fixed point theorem guarantees that F has a fixed point $x \in E$ such that $(Fx)(t) = x(t)$. Hence the theorem.

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