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TOTALLY UMBILICAL CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

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Abstract. In the present paper we study totally umbilical CR-submanifolds of a Kaehler manifold. A classification theorem for a $D^{\perp}$-totally umbilical CR-submanifold is proved. The conditions under which a CR-submanifold becomes a CR-product are obtained, and finally a theorem for a CR-submanifold to be a proper CR-product is also established.

1. Introduction

The notion of a Cauchy-Riemann (CR)-submanifold of a Kaehler manifold was introduced by *A.* Bejancu [1,2]. Subsequently a number of authors studied these, and in particular the totally umbilical CR-submanifold of a Kaehler manifold ([3], [5], [6], [8]).

B. Y. Chen [8] classified the totally umbilical CR-submanifold of a Kaehler manifold and showed that they are either totally geodesic, or totally real, or dim $(D^{\perp}) = 1$. Analogous to this we studied $D^{\perp}$-totally umbilical CR-submanifolds of a Kaehler manifold in §3 and obtained a classification theorem for such submanifolds in a Kaehler setting. In §4 we have investigated the situation under which the CR-submanifolds become a CR-product. Moreover, in the context of Chen's theorem that a complex space form $\overline{M}(c)$ with $c < 0$ does not admit a proper CR-product, it was interesting to establish the existance of a proper CR-product in a complex space form $\overline{M}(c)$ when $c > 0$. Finally we have investigated

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the situation under which the CR-submanifold becomes a proper CR-product in general.

2. Preliminaries

In this section we shall give a brief summary of basic formulas and definitions which are frequently used in the sequel. Let $\overline{M}$ be a Kaehler manifold with almost complex structure J and M be a submanifold of $\overline{M}$. If there exists on M a holomorphic distribution D such that its orthogonal complement $D^\perp$ is totally real in $\overline{M}$ (i.e., $JD_x = D_x$ and $JD_x^\perp \subseteq T_x^\perp M$, $x \in M$), then M is called a CR-submanifold of $\overline{M}$ [1]. If $D = \{0\}$, (resp. $D^\perp = \{0\}$), then M is said to be totally real (resp. holomorphic) submanifold. It follows that the normal bundle $\overset{\perp}{TM}$ splits as $\overset{\perp}{TM} = JD \oplus u$, where u is the orthogonal complement of $JD^\perp$ and is invarient subbundle of $\overset{\perp}{TM}$ under J. Let $\overline{\nabla}$ be the Riemannian connection on $\overline{M}$, then the Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{2.2}$$

for each vector fields X, Y tangent to M and N normal to M, where ∇ is the Riemannian connection of M, h and A are both the second fundamental forms related by

$$g(A_N X, Y) = g(h(X,Y), N), \tag{2.3}$$

and $\nabla^\perp$ the connection in the normal bundle $\overset{\perp}{T(M)}$ of M.

A CR-submanifold M is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H, \tag{2.4}$$

where $H = \frac{1}{n}$ (trace of h), called the mean curvature vector. M is said to be $D^\perp$-totally umbilical (resp. D-totally umbilical) if $h(Z,W) = g(Z,W)H_{D^\perp}$ (resp. $h(X,Y) = g(X,Y)H_D$) holds for every Z, W in $D^\perp$ (X, Y belongs to D)

[9]. M is called $(D^{\perp} - u)$ totally geodesic if $A_N Z = 0$ for each N in u and Z in $D^{\perp}$. For the vector field N normal to M, we put

$$JN = BN + CN, \tag{2.5}$$

where BN and CN are the tangential and normal components of JN.

3. $D^{\perp}$-Totally Umbilical CR-Submanifold

In the present section we shall prove a classification theorem for a $D^{\perp}$-totally umbilical CR-submanifold of a Kaehler manifold. In fact we prove the following:

Theorem 3.1. *Let M be a $D^{\perp}$-totally umbilical CR-submanifold of a Kaehler manifold $\overline{M}$. Then*

(1) *M is $D^{\perp}$-totally geodesic, or*

(2) *M is totally real, or*

(3) *M is proper, and not $(D^{\perp} - u)$ totally geodesic, or*

(4) *the totally real distribution is one-dimensional, i.e.,* $\dim D^{\perp} = 1$.

Proof. We take Z, W in $D^{\perp}$ and using $D^{\perp}$-totally umbilicalness of M together (2.1) and (2.2) with the fact the $\overline{M}$ is Kaehler, we have

$$J\nabla_Z W + g(Z,W)JH_{D^{\perp}} = -A_{JW}Z + \nabla_Z^{\perp} JW.$$

Taking inner product with Z in $D^{\perp}$ it follows that

$$g(H_{D^{\perp}}, JW)\|Z\|^2 = g(Z,W)g(H_{D^{\perp}}, JZ). \tag{3.1}$$

Interchanging Z and W in (3.1) we obtain

$$g(H_{D^{\perp}}, JZ)\|W\|^2 = g(Z,W)g(H_{D^{\perp}}, JW). \tag{3.2}$$

From (3.1) and (3.2), one can immediately get

$$g(H_{D^{\perp}}, JW) = \frac{g(Z,W)^2}{\|Z\|^2\|W\|^2} g(H_{D^{\perp}}, JW). \tag{3.3}$$

The possible solutions of (3.3) are:

(a) $H_{D^\perp} = 0$, or (b) $H_{D^\perp} \perp JW$, or (c) $Z \| W$.

Suppose condition (a) holds, i.e., $H_{D^\perp} = 0$ which implies that M is $D^\perp$-totally geodesic. This proves part (1) of the theorem. Next, suppose that $H_{D^\perp} \neq 0$ which implies $D^\perp \neq 0$ (where D may or may not be zero) swhowing that M is either totally real or proper. Further since $H_{D^\perp}$ belongs to u, there exist a N in u such that $g(h(Z,Z),N) \neq 0$ which implies that $A_N Z \neq 0$ showing that it is not $(D^\perp - u)$ totally geodesic. This proves (2) and (3). Finally if (c) satisfies (3.3), then dim $D^\perp = 1$, which completes the proof of the classification theorem.

From the above theorem, the following is obvious:

Corollary 3.1. *Let M be $D^\perp$-totally umbilical generic CR-submanifold of a Kaehler manifold. Then one of the following holds*

(1) *M is $D^\perp$-totally geodesic*

(2) dim $D^\perp = 1$.

4. CR-Product Submanifolds

In this section we study CR-submanifolds of a Kaehler manifold, and the conditions under which these submanifolds become CR-product. We recall that a CR-submanifold M of a Kaehler manifold $\overline{M}$ is called CR-product if it is locally Riemannian product of a holomorphic submanifold $M^\top$ and a totally real submanifold $M^\perp$ of $\overline{M}$.

B. Y. Chen [6] proved that a CR-submanifold M in a Kaehler manifold $\overline{M}$ is a CR-product if and only if $A_{JD^\perp} D = 0$. Now since for totally umbillical CR-submanifold M of a kaehler manifold with $\dim(D^\perp) \geq 2$, H lies in u [10], as a result M turns out to be a CR-product. Consequently for D-totally umbilical CR-submanifold with $\dim(D^\perp) \geq 2$, D is parallel. In addition to this if M is mixed totally geodesic, the underlying manifold becomes a CR-product. In view of this it is easy to infer:

Lemma 4.1. *Let M be a D-totally umbilical CR-submanifold of a Kaehler manifold $\overline{M}$. Then M is a CR-product if and only if the leaf of $D^{\perp}$ is totally geodesic in M.*

In terms of second fundamental form, the idea of CR-product can be put as:

Proposition 4.1. *Let M be a CR-submanifold of a Kaehler manifold $\overline{M}$. Then M is a CR-product if and only if any one of the following holds.*
(1) $h(U,X) \in u$.
(2) $h(U,JX) = Jh(U,X)$.
for each X in D and U tangent to M.

Proof. The first part follows from the characterization of CR-product given by B. Y. Chen [6], that M is a CR-product if and only if $A_{JD^{\perp}} D = 0$. For the second part, put

$$\begin{aligned} h(U,JX) &= \overline{\nabla}_U JX - \nabla_U JX, \text{ or} \\ h(U,JX) &= J\nabla_U X + Jh(U,X) - \nabla_U JX \end{aligned} \tag{4.1}$$

Form (4.1) it follows that

$$h(U,JX) - Jh(U,X) = J\nabla_U X - \nabla_U JX. \tag{4.2}$$

If M is a CR-product, then the left hand side of (4.2) belongs to u, by part (1) whereas the right hand side belongs to D. Thus both sides are zero which proves that $h(U,JX) = Jh(U,X)$ and the converse is obvious.

Furthermore, for the CR-product submanifold to be proper we have the following result.

Theorem 4.2. *Let M be a D-totally geodisic, but not totally geodesic, CR-submanifold of a Kaehler manifold $\overline{M}$. If the leaf of $D^{\perp}$ is totally geodesic in $\overline{M}$, then M is a proper CR-product. If in addition, M is a complex space form of constant curvature c, then $c > 0$.*

Proof. It is obvious to see that the leaves of D and $D^\perp$ are totally geodesic in M, and hence M is a CR-product. Moreover, using (2.5) and the fact that $\|h(X,Z)\|^2 > 0$ for each $0 \neq X$ in D and $0 \neq Z$ in $D^\perp$ we have

$$g(h(X,Z),\ JB(X,Z)) + g(h(X,Z),\ JCh(X,Z)) < 0. \tag{4.3}$$

Since M is a CR-product, therefore first term of (4.3) vanish and we are left with

$$g(A_{JCh(X,Z)}X, Z) < 0 \tag{4.4}$$

From (4.4) follows that $A_u X$ as well as $A_u Z$ are non zero for each $0 \neq X$ in D and $0 \neq Z$ in $D^\perp$.

On the contrary, suppose M is not proper. Then either $D = 0$ or $D^\perp = 0$.

Case 1. Suppose $D^\perp = 0$ but $D \neq 0$. Then $A_N X$ does not belong to $D^\perp$ for each X in D and N in $\overset{\perp}{TM}$, which contradicts the fact that M is D-totally geodesic. Similarly,

Case 2. Suppose $D = 0$ but $D^\perp \neq 0$. Then $A_N Z$ does not belong to D for each Z in $D^\perp$ and N in $\overset{\perp}{TM}$, which again contradicts the fact that M is $D^\perp$-totally geodesic. Hence neither D nor $D^\perp$ can be zero, which completes the proof of the theorem.

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