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GENERALIZED CLASS OF UNIVALENT FUNCTIONS WITH TWO FIXED POINTS

B. A. URALEGADDI AND C. SOMANATHA

Abstract. Univalent functions of the form $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$, where $a_n \geq 0$, are dealt with. We examine the subclasses for which $(1-\lambda)f(z_0)/z_0 + \lambda f'(z_0) = 1$ $(-1 < z_0 < 1)$. The coefficient inequalities and the extreme points of the classes that are starlike and convex of order α are determined. Many of the results of Silverman are obtained as particular cases.

1. Introduction

In [1], Silverman examined the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$$

that are univalent in the unit disk $\Delta = \{z : |z| < 1\}$. In [2], silverman studied the class of univalent functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0$$

where either

$$f(z_0) = z_0, \quad (-1 < z_0 < 1;\ z_0 \neq 0)$$

or

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$$f'(z_0) = 1 \quad (-1 < z_0 < 1).$$

In this paper we consider functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$$

where

$$a_n \geq 0,\ (1-\lambda)f(z_0)/z_0 + \lambda f'(z_0) = 1 \quad (-1 < z_0 < 1;\ 0 \leq \lambda \leq 1). \tag{1}$$

A function $f(z)$ is said to be starlike of order α, $0 \leq \alpha < 1$, if

$$\text{Re}\{zf'(z)/f(z)\} > \alpha \quad (|z| < 1)$$

and is said to be convex of order α if

$$\text{Re}\{1 + zf''(z)/f'(z)\} > \alpha \quad (|z| < 1).$$

Given α and z_0 fixed let $S^*_\lambda(\alpha, z_0)$ and $K_\lambda(\alpha, z_0)$ be the subclasses of functions starlike of order α and convex of order α respectively satisfying (1). For these classes we obtain coefficient estimates, comparable results and extreme points.

The special cases of $S^*_\lambda(\alpha, z_0)$ and $K_\lambda(\alpha, z_0)$ are the subclasses $S^*_0(\alpha, z_0)$, $S^*_1(\alpha, z_0)$, $K_0(\alpha, z_0)$ and $K_1(\alpha, z_0)$ respectively. Several results on these subclasses may be found in [2].

We need the following results due to Silverman [2].

Theorem A. *A function* $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ *is starlike of order* α *if and only if* $\sum_{n=2}^{\infty} (n-\alpha)a_n \leq a_1(1-\alpha)$.

Theorem B. *A function* $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ *is convex of order* α *if and only if* $\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq a_1(1-\alpha)$.

2. Coefficient Inequalities

Theorem 1. *Suppose* $a_n \geq 0$ *for every* n. *Then* $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ *is in* $S_\lambda^*(\alpha, z_0)$ *if and only if* $\sum_{n=2}^{\infty} a_n[\frac{(n-\alpha)}{(1-\alpha)} - ((1-\lambda) + n\lambda)z_0^{n-1}] \leq 1$.

Proof. Since

$$\begin{aligned} &(1-\lambda)f(z_0)/z_0 + \lambda f'(z_0) \\ &= (1-\lambda)(a_1 - \sum_{n=2}^{\infty} a_n z_0^{n-1}) + \lambda(a_1 - \sum_{n=2}^{\infty} n a_n z_0^{n-1}) \\ &= a_1 - \sum_{n=2}^{\infty} a_n[(1-\lambda) + n\lambda]z_0^{n-1} = 1, \end{aligned}$$

we have

$$a_1 \;=\; 1 + \sum_{n=2}^{\infty} a_n[(1-\lambda) + n\lambda]z_0^{n-1}. \tag{2}$$

Substituting this value of a_1 in the statement of Theorem A we get

$$\sum_{n=2}^{\infty} a_n[\frac{(n-\alpha)}{(1-\alpha)} - ((1-\lambda) + n\lambda)z_0^{n-1}] \leq 1.$$

Corollary 1. *If* $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ *is in* $S_\lambda^*(\alpha, z_0)$ *then*

$$a_n \leq (1-\alpha)/[(n-\alpha) - (1-\alpha)((1-\lambda) + n\lambda)z_0^{n-1}] \quad (n = 2, 3, \ldots)$$

with equality for

$$f(z) \;=\; \frac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)((1-\lambda) + n\lambda)z_0^{n-1}}.$$

Theorem 2. *Let* $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ *and satisfy* (1). *Then* $f(z)$ *is univalent if and only if* $\sum_{n=2}^{\infty} a_n[n - ((1-\lambda) + n\lambda)z_0^{n-1}] \leq 1$.

Proof. Since $a_1 = 1 + \sum_{n=2}^{\infty} a_n[(1-\lambda) + n\lambda]z_0^{n-1}$, it suffices to show that $f(z)$ is univalent if and only if $\sum_{n=2}^{\infty} na_n \le a_1$.

Suppose $\sum_{n=2}^{\infty} na_n > a_1$. We can write $\sum_{n=2}^{\infty} na_n = a_1 + \varepsilon$, $(\varepsilon > 0)$. Then there exists an integer N such that

$$\sum_{n=2}^{N} na_n > a_1 + \frac{\varepsilon}{2}.$$

For z in the interval $(a_1/(a_1 + \frac{\varepsilon}{2}))^{\frac{1}{(N-1)}} < z < 1$, we have

$$\begin{aligned} f'(z) &\le a_1 - \sum_{n=2}^{N} na_n z^{n-1} \\ &\le a_1 - z^{N-1} \sum_{n=2}^{N} na_n \\ &< a_1 - z^{N-1}(a_1 + \frac{\varepsilon}{2}) < 0. \end{aligned}$$

Since $f'(0) > 0$, there exists a real number z_0, $0 < z_0 < 1$, such that $f'(z_0) = 0$. Hence f is not univalent.

Conversely, let $\sum_{n=2}^{\infty} a_n[n - ((1-\lambda) + n\lambda)z_0^{n-1}] \le 1$. Then by Theorem 1, $f \in S_\lambda^*(0, z_0)$. Hence f is univalent.

Theorem 3. *Suppose $a_n \ge 0$ for every n. Then $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ is in $K_\lambda(\alpha, z_0)$ if and only if $\sum_{n=2}^{\infty} a_n[\frac{n(n-\alpha)}{(1-\alpha)} - ((1-\lambda) + n\lambda)z_0^{n-1}] \le 1$.*

Proof. Substituting the value of a_1 given by (2), in the statement of Theorem B the result follows.

Corollary 2. *If $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ is in $K_\lambda(\alpha, z_0)$ then*

$$a_n \le (1-\alpha)/[n(n-\alpha) - (1-\alpha)((1-\lambda) + n\lambda)z_0^{n-1}] \qquad (n = 2, 3, \dots)$$

with equality for

$$f(z) = \frac{n(n-\alpha)z-(1-\alpha)z^n}{n(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}}$$

3. Comparable Results

Theorem 4. *If $f(z) \in K_\lambda(\alpha, z_0)$ then $f(z) \in S_\lambda^*(2/(3-\alpha), z_0)$.*

Proof is similar to that of Theorem 3 in [2]. The result is sharp, with extremal function

$$f(z) = \frac{2(2-\alpha)z-(1-\alpha)z^2}{2(2-\alpha)-(1-\alpha)((1-\lambda)+2\lambda)z_0}$$

Theorem 5. *If $f(z) \in S_\lambda^*(\alpha, z_0)$ then $f(z)$ is convex in the disk*

$$|z| < r = r(\alpha) = \inf_n \left[\frac{n-\alpha}{n^2(1-\alpha)}\right]^{\frac{1}{n-1}} \qquad n = 2, 3, \ldots$$

The result is sharp for the extremal function

$$f_n(z) = \frac{(n-\alpha)z-(1-\alpha)z^n}{(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}} \qquad (n = 2, 3, \ldots).$$

Proof. It suffices to show that

$$|z\frac{f''(z)}{f'(z)}| \le 1 \quad \text{for} \quad |z| < r(\alpha).$$

We have

$$|z\frac{f''(z)}{f'(z)}| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}}$$

Thus

$$|z\frac{f''(z)}{f'(z)}| \le 1 \quad \text{if}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \le a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}$$
$$= 1 + \sum_{n=2}^{\infty} a_n((1-\lambda)+n\lambda)z_0^{n-1} - \sum_{n=2}^{\infty} na_n|z|^{n-1}$$

that is if

$$\sum_{n=2}^{\infty} a_n[n^2|z|^{n-1} - ((1-\lambda)+n\lambda)z_0^{n-1}] \le 1. \tag{3}$$

From Theorem 1, we have

$$\sum_{n=2}^{\infty} a_n[\frac{(n-\alpha)}{(1-\alpha)} - ((1-\lambda)+n\lambda)z_0^{n-1}] \le 1.$$

Hence (3) will be true if

$$n^2|z|^{n-1} - ((1-\lambda)+n\lambda)z_0^{n-1} \le \frac{(n-\alpha)}{(1-\alpha)} - ((1-\lambda)+n\lambda)z_0^{n-1} \tag{4}$$

solving (4) for $|z|$ we get

$$|z| \le \left[\frac{n-\alpha}{n^2(1-\alpha)}\right]^{\frac{1}{n-1}} \quad n = 2, 3, \ldots,$$

the result follows.

Remark. The conclusions in Theorem 4 and 5 are independent of λ and the fixed point z_0.

4. Extreme Points

Theorem 6. *The extreme points of $S^*_\lambda(\alpha, z_0)$ are given by $f_1(z) = z$ and* $f_n(z) = \dfrac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}}$ $(n = 2, 3, \ldots)$.

Proof. It suffices to show that $f(z) \in S^*_\lambda(\alpha, z_0)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Suppose $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$. Then

$$\begin{aligned} f(z) &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\ &= \left[\lambda_1 + \sum_{n=2}^{\infty} \frac{\lambda_n(n-\alpha)}{(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}}\right] z \\ &\quad - \sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha)z^n}{(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}}. \end{aligned}$$

Note that

$$\begin{aligned} &[(1-\lambda)f(z_0)/z_0] + \lambda f'(z_0) \\ &= [(1-\lambda)\lambda_1 f_1(z_0)/z_0] + (1-\lambda)\sum_{n=2}^{\infty} \lambda_n f_n(z_0)/z_0 \\ &\quad + \lambda\lambda_1 f_1'(z_0) + \lambda \sum_{n=2}^{\infty} \lambda_n f_n'(z_0). \\ &= \lambda_1 + \sum_{n=2}^{\infty} \lambda_n = 1. \end{aligned}$$

Also we see that

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha)}{(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}} \left[\frac{(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}}{1-\alpha}\right] \\ &= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1. \end{aligned}$$

Therefore from Theorem 1, $f(z) \in S_\lambda^*(\alpha, z_0)$.

Conversely, if $f(z) \in S_\lambda^*(\alpha, z_0)$, then

$$a_n \leq (1-\alpha)/[(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}] \quad (n = 2, 3, \ldots).$$

Set

$$\lambda_n = \{[(n-\alpha)-(1-\alpha)((1-\lambda)+n\lambda)z_0^{n-1}]/(1-\alpha)\}a_n \quad (n = 2, 3, \ldots)$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$.

Similarly, the coefficients bounds on $K_\lambda(\alpha, z_0)$ enable us to prove

Theorem 7. *The extreme points of* $K_\lambda(\alpha, z_0)$ *are given by* $f_1(z) = z$ *and*

$$f_n(z) = \frac{n(n-\alpha)z - (1-\alpha)z^n}{n(n-\alpha) - (1-\alpha)((1-\lambda) + n\lambda)z_0^{n-1}} \qquad (n = 2, 3, \ldots).$$

5. Convex Families

Suppose B is nonempty subset of the real interval (0,1), we define $S_\lambda^*(\alpha, B)$ by

$$S_\lambda^*(\alpha, B) = \bigcup_{z_\gamma \in B} S_\lambda^*(\alpha, z_\gamma).$$

If B consists of a single element say z_0 then $S_\lambda^*(\alpha, z_0)$ is a convex family. Because if $f_1(z)$ and $f_2(z)$ are in $S_\lambda^*(\alpha, z_0)$, then it can be seen that for $0 \le \delta \le 1$, $f(z) = \delta f_1(z) + (1-\delta) f_2(z)$ is in $S_\lambda^*(\alpha, z_0)$. Next we examine this class for other subsets of B. We need the following.

Lemma. *If* $f(z) \in S_\lambda^*(\alpha, z_0) \cap S_\lambda^*(\alpha, z_1)$, *where* z_0 *and* z_1 *are distinct positive numbers then* $f(z) = z$.

Proof. Taking $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \ge 0)$, we have

$$\begin{aligned} a_1 &= 1 + \sum_{n=2}^{\infty} a_n[(1-\lambda) + n\lambda] z_0^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} a_n[(1-\lambda) + n\lambda] z_1^{n-1} \end{aligned}$$

That is

$$a_n[(1-\lambda) + n\lambda][z_1^{n-1} - z_0^{n-1}] = 0.$$

Hence $a_n \equiv 0$ for $n \ge 2$, and the result follows.

Theorem 8. *If B is contained in the interval $(0,1)$ and $0 \leq \alpha < 1$, then $S^*_\lambda(\alpha, B)$ is a convex family if and only if B is connected.*

Proof. Let B be connected. Suppose z_0, $z_1 \in B$ with $z_0 \leq z_1$. If $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ is in $S^*_\lambda(\alpha, z_0)$ and $g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n$ is in $S^*_\lambda(\alpha, z_1)$ then for $0 \leq \delta \leq 1$, we shall prove that there exists a $z_2 (z_0 \leq z_2 \leq z_1)$ such that $h(z) = \delta f(z) + (1-\delta) g(z)$ is in $S^*_\lambda(\alpha, z_2)$. Set

$$\begin{aligned} t(z) &= [(1-\lambda)h(z)/z] + \lambda h'(z) \\ &= \delta\{a_1 - \sum_{n=2}^{\infty} a_n z^{n-1}((1-\lambda) + n\lambda)\} \\ &\quad + (1-\delta)\{b_1 - \sum_{n=2}^{\infty} b_n z^{n-1}((1-\lambda)+n\lambda)\} \\ t(z) &= 1 + \delta \sum_{n=2}^{\infty} a_n\{(1-\lambda)+n\lambda\}(z_0^{n-1} - z^{n-1}) \qquad (5) \\ &\quad + (1-\delta)\sum_{n=2}^{\infty} b_n\{(1-\lambda)+n\lambda\}(z_1^{n-1} - z^{n-1}) \end{aligned}$$

and we observe that $t(z)$ is real when z is real with $t(z_0) \geq 1$ and $t(z_1) \leq 1$. Hence for some z_2, $z_0 \leq z_2 \leq z_1$, we have $t(z_2) = 1$. Since z_1, z_2 and δ are arbitrary, the family $S^*_\lambda(\alpha, B)$ is convex.

Conversely, suppose B is not connected. Then we can take z_0, $z_1 \in B$, $z_2 \notin B$ such that $z_0 < z_2 < z_1$. Let us assume $f(z)$ and $g(z)$ are not both identity function. Then using (5) and fixing $z = z_2$ and allow δ to vary,

$$\begin{aligned} t(\delta) &= t(z_2, \delta) \\ &= 1 + \delta \sum_{n=2}^{\infty} a_n[(1-\lambda)+n\lambda](z_0^{n-1} - z_2^{n-1}) \\ &\quad + (1-\delta)\sum_{n=2}^{\infty} b_n[(1-\lambda)+n\lambda](z_1^{n-1} - z_2^{n-1}). \end{aligned}$$

Since $t(z_2, 0) > 1$ and $t(z_2, 1) < 1$, there must exists a δ_0, $0 < \delta_0 < 1$, for which $t(z_2, \delta_0) = 1$. Hence $h(z) \in S^*_\lambda(\alpha, z_2)$ for $\delta = \delta_0$. Since $z_2 \notin B$ from the above lemma it follows that $h(z) \notin S^*_\lambda(\alpha, B)$. Therefore $S^*_\lambda(\alpha, B)$ is not a convex family.

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Department of Mathematics, Karnatak University, Dharwad-580003, India.