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ON THE SCHWARZIAN COEFFICIENTS OF THE KOEBE TRANSFORM OF A UNIVALENT FUNCTION

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Abstract. For $f \in S$, we compute the Schwarzian coefficients of the Koebe Transform f_{ϕ_α} of f in terms of successive derivatives of the Schwar- zian derivative, then provide some estimates on certain combinations of them.

Let S denote the class of functions $f(z) = z + a_2z^2 + \ldots$ which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$.

The *Schwarzian derivative* of a function $f(z)$ in S is defined by the relation

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1}$$

and the *Schwarzian coefficients* of the function $f(z)$ are the Taylor coefficients in the series expansion

$$\{f, z\} = \sum_{n=0}^{\infty} s_n z^n \tag{2}$$

In a previous paper, the author [14] determined sharp estimates on the coefficients s_0, s_1 and s_2, as well as a general estimate on s_n, and general estimates on certain linear combinations of Schwarzian coefficients. In this paper, we determine explicit formulas for the Schwarzian coefficients of the Koebe Transform of a function $f \in S$ in terms of the successive derivatives of the Schwarzian derivative, and then provide growth estimates on certain combinations of them.

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Let $w(z)$ be an analytic, univalent function in U with $|w(z)| < 1$. If $f \in S$, then

$$f_w(z) = \frac{f(w(z)) - f(w(0))}{f'(w(0))w'(0)}$$

also belongs to S. The Schwarzian derivatives of f and f_w, and hence their coefficients, are related by the composition law [4, p.376]

$$\{f_w, z\} = \{f, w(z)\}(w'(z))^2 + \{w, z\}. \tag{3}$$

If we now choose

$$w(z) = \phi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z} = -\alpha + (1-|\alpha|^2)\sum_{m=1}^{\infty} \overline{\alpha}^{m-1} z^m, \ (\alpha \in U) \tag{4}$$

then $f_w = f_{\phi_\alpha} \in S$ is known as the *Koebe Transform of* f, and the composition law becomes

$$\{f_{\phi_\alpha}, z\} = \{f, \phi_\alpha(z)\}(\phi'_\alpha(z))^2.$$

Let us investigate the Schwarzian coefficients of f_{ϕ_α}. If we write, for $n \geq 0$,

$$(\phi_\alpha(z))^n = \sum_{m=0}^{\infty} c_m^{(n)}(\alpha) z^m, \tag{5}$$

then

$$\{f, \phi_\alpha(z)\} = \sum_{n=0}^{\infty} s_n(f)(\phi_\alpha(z))^n = \sum_{m=0}^{\infty} C_m(f;\alpha) z^m$$

where

$$C_0(f;\alpha) = \sum_{n=0}^{\infty} s_n(f) c_0^{(n)}(\alpha)$$

and

$$C_m(f;\alpha) = \sum_{n=1}^{\infty} s_n(f) c_m^{(n)}(\alpha). \quad (m \geq 1) \tag{6}$$

Since

$$(\phi'_\alpha(z))^2 = (1-|\alpha|^2)^2 \sum_{m=0}^{\infty} \binom{m+3}{3} \overline{\alpha}^m z^m,$$

a comparison of coefficients in the composition law shows that the Schwarzian coefficients of the Koebe Transform are given by

$$s_n(f_{\phi_\alpha}) = (1-|\alpha|^2)^2 \sum_{m=0}^{n} \binom{n+3-m}{3} \overline{\alpha}^{n-m} C_m(f;\alpha). \tag{7}$$

In order to estimate these coefficients and linear combinations of them, we must first determine the sums $C_m(f;\alpha)(m = 0, 1, 2, \dots)$.

Lemma. *Let $f \in S$ and let $\{f,z\}^{(r)}(r = 1, 2, 3, \dots)$ denote the rth derivative of $\{f, z\}$. Then,*

$$C_0(f;\alpha) = \{f, -\alpha\}$$

and, for $m \geq 1$,

$$C_m(f;\alpha) = \sum_{r=1}^{m} \frac{1}{r!} \binom{m-1}{r-1} (1-|\alpha|^2)^r \overline{\alpha}^{m-r} \{f, -\alpha\}^{(r)}. \tag{8}$$

Proof. We must first determine the coefficients $c_m^{(n)}(\alpha)$ defined by (5). It is clear from (4) that $c_0^{(0)}(\alpha) = 1$, given by $c_m^{(0)}(\alpha) = 0$ $(m = 1, 2, \dots)$, and that $c_0^{(n)}(\alpha) = (-\alpha)^n (n = 1, 2, 3, \dots)$. Clearly, $C_0(f;\alpha) = \sum_{n=0}^{\infty} s_n(f)(-\alpha)^n = \{f, -\alpha\}$. For all n, $m \geq 1$, a tedious induction on n, using the relation given by $c_m^{(n+1)}(\alpha) = \sum_{j=0}^{m} c_j^{(n)}(\alpha) \cdot c_{m-j}^{(1)}(\alpha)$, shows that

$$c_m^{(n)}(\alpha) = \sum_{r=1}^{m} \binom{m-1}{r-1} \binom{n}{r} (1-|\alpha|^2)^r \overline{\alpha}^{m-r} (-\alpha)^{n-r}, \tag{9}$$

where it is understood that $\binom{n}{r} = 0$, if $r > n$. Substituting (9) into (6) and carefully interchanging sums establishes (8) for all $m \geq 1$.

In [14], the author established the sharp inequalities $|s_0| \leq 6$, $|s_1| \leq 16$ and $|s_2| \leq 30(1 + 2e^{-34/3})$. As of a consequence of these estimates and the above lemma, we may obtain *sharp* bounds on the quantities

$$s_0(f_{\phi_\alpha}) = (1-|\alpha|^2)^2 \{f, -\alpha\}$$
$$s_1(f_{\phi_\alpha}) = (1-|\alpha|^2)^2 [4\overline{\alpha}\{f, -\alpha\} + (1-|\alpha|^2)\{f, -\alpha\}^{(1)}]$$

and

$$s_2(f_{\phi_\alpha}) = (1-|\alpha|^2)^2[10\overline{\alpha}^2\{f,-\alpha\} + 5\overline{\alpha}(1-|\alpha|^2)\{f,-\alpha\}^{(1)} + \frac{1}{2}(1-|\alpha|^2)^2\{f,-\alpha\}^{(2)}]$$

for any $\alpha \in U$. Also, an application of the triangle inequality now shows that

$$|\{f,z\}^{(1)}| \leq \frac{16+24|z|}{(1-|z|^2)^3},$$

which is sharp if $z=0$, and has the correct order of growth for $|z|$ near 1. This inequality may be used in turn to provide an estimate for $|\{f,z\}^{(2)}|$.

We are now ready to state our general result. Here, $(a)_r$ represents the Appell symbol defined by $(a)_0 = 1$ and $(a)_r = a(a+1)\ldots(a+r-1)$ for $r \geq 1$.

Theorem. *Let $f \in S$ and let f_{ϕ_α} denote the Koebe Transform of f. Then, the Schwarzian coefficients of f_{ϕ_α} are given by*

$$s_n(f_{\phi_\alpha}) = (1-|\alpha|^2)^2\binom{n+3}{3}\sum_{r=0}^{n}\frac{(-1)^r(-n)_r}{(4)_r(1)_r}(1-|\alpha|^2)^r\overline{\alpha}^{n-r}\{f,-\alpha\}^{(r)} \quad (10)$$

for all $n \geq 0$, where $\{f,-\alpha\}^{(r)}$ $(r=0,1,2,\ldots)$ denotes the rth derivative of $\{f,z\}$ evaluated at $z=-\alpha$. Consequently, for every $N \geq 0$, we have

$$\left|\sum_{r=0}^{N}\frac{1}{(4)_r(1)_r}P(N,r;\overline{\alpha})(1-|\alpha|^2)^r\{f,-\alpha\}^{(r)}\right| \leq \frac{(N+1)_3}{(1-|\alpha|^2)^2} \quad (11)$$

where

$$P(N,r;\overline{\alpha}) = {}_2F_1(-N+r, r+4; -N+r-1; \overline{\alpha}).$$

is a hypergeometric polynomial of degree $(N-r)$ in the variable $\overline{\alpha}$.

Proof. Substituting (8) into (7), and then interchanging sums, we obtain

$$\begin{aligned}\frac{s_n(f_{\phi_\alpha})}{(1-|\alpha|^2)^2} &= \binom{n+3}{3}\overline{\alpha}^n\{f,-\alpha\} \\ &\quad + \sum_{r=1}^{n}\left[\sum_{m=r}^{n}\binom{n+3-m}{3}\binom{m-1}{r-1}\right]\frac{1}{r!}(1-|\alpha|^2)^r\overline{\alpha}^{n-r}\{f,-\alpha\}^{(r)} \\ &= \binom{n+3}{3}\overline{\alpha}^n\{f,-\alpha\} \\ &\quad + \sum_{r=1}^{n}\binom{n+3}{3}\frac{(-1)^r(-n)_r}{(4)_r(1)_r}(1-|\alpha|^2)^r\overline{\alpha}^{n-r}\{f,-\alpha\}^{(r)}\end{aligned}$$

which establishes (10). (The inner sum was evaluated by using Vandermonde's formula, after a shift of index.) Hence, after interchanging sums, we get

$$\sum_{n=0}^{N}(N+1-n)s_n(f_{\phi_\alpha}) = (1-|\alpha|^2)^2\sum_{r=0}\frac{1}{(4)_r(1)_r}P(N,r;\overline{\alpha})(1-|\alpha|^2)^r\{f,-\alpha\}^{(r)}$$

where

$$\begin{aligned} P(N,r;\overline{\alpha}) &= \sum_{n=r}^{N}(N+1-r)\binom{n+3}{3}(-1)^r(-n)_r\overline{\alpha}^{n-r} \\ &= \sum_{n=0}^{N-r}(N+1-n-r)\binom{n+r+3}{3}(-n-r)_r\overline{\alpha}^n \\ &= \sum_{n=0}^{N-r}p_n\overline{\alpha}^n \end{aligned}$$

Since

$$\frac{p_{n+1}}{p_n} = \frac{(-N+r+n)(r+4+n)}{(-N+r-1+n)(1+n)},$$

it is clear that

$$p_n = \frac{(-N+r)_n(r+4)_n}{(-N+r-1)_n}\cdot\frac{1}{n!}$$

so that $P(N,r;\overline{\alpha})$ is a hypergeometric polynomial of degree $(N-r)$ in $\overline{\alpha}$, as we have claimed. The conclusion (11) now follows directly from the inequality

$$\left|\sum_{n=0}^{N}(N+1-n)s_n\right| \leq (N+1)(N+2)(N+3)$$

which was established in [14, Theorem 3].

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