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A REFINEMENT OF HADAMARD'S INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS

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A refinement of Hadamard's inequality for isotonic linear functionals and some applications to norm and discrete inequalities are given.

1. Introduction

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. The following double inequality

$$f(\frac{x+y}{2}) \leq \frac{1}{y-x}\int_x^y f(t)dt \leq \frac{f(x)+f(y)}{2} \tag{1.1}$$

where $x, y \in I$, $x < y$, is known in literature as Hadamard's inequality (see [6], [9] or [5]). For some recent results in connection with this famous integral inequality we refer to [2-5] and [9-11] where further applications are given.

In this paper we will give an analogous of this fact for isotonic linear functionals (compare with [8]). Some natural applications are also pointed out.

As in [1], let E be a nonempty set and let L be a linear class of real valued functions $g: E \to \mathbb{R}$ having the properties:

$L1: f, g \in L$ imply $(af + bg) \in L$ for all $a, b \in \mathbb{R}$;

$L2: 1 \in L$, that is, if $f(t) = 1$ $(t \in E)$ then $f \in L$.

We also consider isotonic linear functionals $A: L \to \mathbb{R}$. That is, we suppose:

$A1: A(af + bg) = aA(f) + bA(g)$ for all $f, g \in L$ and $a, b \in \mathbb{R}$;

$A2: f \in L, f(t) \geq 0$ on E implies $A(f) \geq 0$.

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We note that common examples of such isotonic linear functionals A are given by $A(g) := \int_E g\, d\mu$ or $A(g) = \sum_{k\in E} p_k g_k$ where μ is a positive measure on E in the first case and E is a subset of natural number $\mathbb{N}$ in the second case with $p_k \geq 0$ for $k \in E$.

We also will use Jensen's inequality (see e.g. [1]):

Theorem 1.1. *Let L satisfy properties $L1$, $L2$ on a nonempty set E and suppose ϕ is a convex function on an interval $I \subseteq \mathbb{R}$. If A is any isotonic linear functional with $A(1) = 1$, then for all $g \in L$ so that $\phi(g) \in L$, we have $A(g) \in I$ and*

$$\phi(A(g)) \leq A(\phi(g)).$$

2. The Main Result

We will start with the following simple lemma.

Lemma 2.1. *Let X be a real linear space and C be its convex subset. If $f : C \to \mathbb{R}$ is convex on C, then for all x, y in C the mapping $g_{x,y} : [0,1] \to \mathbb{R}$ given by*

$$g_{x,y}(t) := 1/2\big[f(tx + (1-t)y) + f((1-t)x + ty)\big]$$

is also convex on $[0,1]$. In addition, we have the inequality:

$$f\Big(\frac{x+y}{2}\Big) \leq g_{x,y}(t) \leq \frac{f(x) + f(y)}{2} \tag{2.1}$$

for all x, y in C and $t \in [0,1]$.

Proof. Suppose $x, y \in C$ and let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned}
&g_{x,y}(\alpha t_1 + \beta t_2) \\
&= \frac{1}{2}\left(f\left[(\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2)y\right] + f\left[(1 - \alpha t_1 - \beta t_2)x + (\alpha t_1 + \beta t_2)y\right]\right) \\
&= \frac{1}{2}\left(f\left[\alpha(t_1 x + (1-t_1)y) + \beta(t_2 x + (1-t_2)y\right]\right. \\
&\quad \left. + f\left[\alpha((1-t_1)x + t_1 y) + \beta((1-t_2)x + t_2 y)\right]\right)
\end{aligned}$$

$$\leq \frac{1}{2}\left(\alpha f\left[t_1x+(1-t_1)y\right]+\beta f\left[t_2x+(1-t_2)y\right]\right.$$
$$\left.+\alpha f\left[(1-t_1)x+t_1y\right]+\beta f\left[(1-t_2)x+t_2y\right]\right)$$
$$= \alpha g_{x,y}(t_1)+\beta g_{x,y}(t_2)$$

which shows that $g_{x,y}$ is convex on [0,1].

By the convexity of f we can state:

$$g_{x,y}(t) \geq f\left[\frac{1}{2}(tx+(1-t)y+(1-t)x+ty)\right] = f\left(\frac{x+y}{2}\right)$$

and also

$$g_{x,y}(t) \leq \frac{1}{2}\left[tf(x)+(1-t)f(y)+(1-t)f(x)+tf(y)\right] = \frac{f(x)+f(y)}{2}$$

for all t in [0,1], which completes the proof.

Remark 2.2. By the inequality (2.1) we can state:

$$\sup_{t\in[0,1]} g_{x,y}(t) = \frac{f(x)+f(y)}{2} \text{ and } \inf_{t\in[0,1]} g_{x,y}(t) = f\left(\frac{x+y}{2}\right)$$

for all x, y in C.

Now, we can give our main result.

Theorem 2.3. *Let $f : C \subseteq X \to \mathbb{R}$ be a convex function on convex set C, L and A satisfy conditions $L1$, $L2$ and $A1$, $A2$ and $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$ $(t \in E)$, $h \in L$ is so that $f(hx+(1-h)y)$, $f((1-h)x+hy)$ belong to L for x, y fixed in C. If $A(1) = 1$, then we have the inequality:*

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq \frac{1}{2}\left[f(A(h)x+(1-A(h))y)+f((1-A(h))x+A(h)y)\right] \\ &\leq \frac{1}{2}\left(A\left[f(hx+(1-h)y)\right]+A\left[f((1-h)x+hy)\right]\right) \\ &\leq \frac{f(x)+f(y)}{2}. \end{aligned} \tag{2.2}$$

Proof. Let consider the mapping $g_{x,y} : [0,1] \to \mathbb{R}$ given above. Then by the above lemma we know that $g_{x,y}$ is convex on [0,1]. Applying Jessen's inequality for the mapping $g_{x,y}$ we get

$$g_{x,y}(A(h)) \leq A(g_{x,y}(h)).$$

But

$$g_{x,y}(A(h)) = \frac{1}{2}\left[f(A(h)x + (1 - A(h))y) + f((1 - A(h))x + A(h)y)\right]$$

and

$$A(g_{x,y}(h)) = \frac{1}{2}\left(A\left[f(hx + (1 - h)y\right] + A\left[f((1 - h)x + hy)\right]\right)$$

and the second inequality in (2.2) is proved.

To prove the first inequality in (2.2) we observe that, by (2.1), we can write

$$f\left(\frac{x+y}{2}\right) \leq g_{x,y}(A(h))$$

which is ecxactly the desired statement.

Finally, we observe that, by the convexity of f, we get

$$\frac{1}{2}\left[f(hx + (1 - h)y) + f((1 - h)x + hy)\right] \leq \frac{f(x) + f(y)}{2} \text{ on } E.$$

Applying to this inequality the functional A and since $A(1) = 1$, we obtain the last part of (2.2).

Remark 2.4. If we chose: $A = \int_0^1$, $E = [0,1]$, $h(t) = t$, $C = [x,y] \subset \mathbb{R}$ and since a simple calculation shows that

$$\int_0^1 f(tx + (1 - t)y)dt = \int_0^1 f((1 - t)x + ty)dt = \frac{1}{y - x}\int_x^y f(t)dt$$

we recapture, by (2.2) the inequality (1.1) of Hadamard.

3. Applications

1. Let $h : [0,1] \to [0,1]$ be a Riemann integrable function on [0,1] and $p \geq 1$. Then for all x, y vectors in normed space $(X; \|\cdot\|)$ we have the inequality:

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|^p &\leq \frac{1}{2}\Big[\|(1 - \int_0^1 h(t)dt)x + (\int_0^1 h(t)dt)y\|^p \\ &\quad + \|(\int_0^1 h(t)dt)x + (1 - \int_0^1 h(t)dt)y\|^p\Big] \qquad (3.1) \\ &\leq \frac{1}{2}\left[\int_0^1 \|h(t)x + (1 - h(t))y\|^p dt + \int_0^1 \|(1 - h(t))x + h(t)y\|^p dt\right] \\ &\leq \frac{\|x\|^p + \|y\|^p}{2} \end{aligned}$$

If we chose $h(t) = t$, we obtain

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|tx + (1-t)y\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2} \tag{3.2}$$

for all x, y in X.

The inequality (3.1) follows by Theorem 2.3 for the functional $A = \int_0^1$.

2. Let $f : C \subseteq X \to \mathbb{R}$ be a convex function on convex set C of a linear space X, $t_i \in [0,1]$, $i = 1, \ldots, n$. Then we have the inequality:

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq \frac{1}{2}\left(f\left[\frac{1}{n}\sum_{i=1}^{n} t_i x + \frac{1}{n}\sum_{i=1}^{n}(1-t_i)y\right] + f\left[\frac{1}{n}\sum_{i=1}^{n}(1-t_i)x + \frac{1}{n}\sum_{i=1}^{n} t_i y\right]\right) \\ &\leq \frac{1}{2n}\left[\sum_{i=1}^{n} f(t_i x + (1-t_i)y) + \sum_{i=1}^{n} f((1-t_i)x + t_i y)\right] \\ &\leq \frac{f(x)+f(y)}{2} \end{aligned} \tag{3.3}$$

If we put $t_i = \sin^2 \alpha_i$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, then we obtain

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq \frac{1}{2}\left(f\left[(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i)x + (\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i)y\right]\right. \\ &\qquad \left. + f\left[(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i)x + (\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i)y\right]\right) \\ &\leq \frac{1}{2n}\sum_{i=1}^{n}\left(f\left[(\sin^2\alpha_i)x + (\cos^2\alpha_i)y\right] + f\left[(\cos^2\alpha_i)x + (\sin^2\alpha_i)y\right]\right) \\ &\leq \frac{f(x)+f(y)}{2} \end{aligned}$$

The inequaltity (3.3) follows by (2.2) for $A = 1/n \sum_{i=1}^{n}$, $h(i) = t_i \in [0,1]$.

By the use of the inequality (3.3), we can obtain a refinement of the arithmetic mean-geometric mean inequality

$$\frac{x+y}{2} \geq \sqrt{xy} \text{ where } x, y \geq 0.$$

Indeed, chosing $f(x) := -\ln x$, $x > 0$, we obtain

$$\frac{x+y}{2} \geq \left[\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right) x + \frac{1}{n}\sum_{i=1}^{n}(1-t_i)y\right]^{1/2} \left[\frac{1}{n}\sum_{i=1}^{n}(1-t_i)x + \left(\frac{1}{n}\sum_{i=1}^{n} t_i\right) y\right]^{\frac{1}{2}}$$

$$\geq \prod_{i=1}^{n}\left([t_i x + (1-t_i)y]^{1/2}\,[(1-t_i)x + t_i y]^{1/2}\right)^{1/n} \geq \sqrt{xy}.$$

The equality holds in the previous inequalities simultaneously iff $x = y$.

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