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ALTERNATION THEOREM FOR $C(I, X)$ AND APPLICATION TO BEST LOCAL APPROXIMATION

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Abstract. Let X be a Banach space with the approximation property, and $C(I, X)$ the space of continuous functions defined on $I = [0, 1]$ with values in X. Let $u_i \in C(I, X)$, $i = 1, 2, \cdots, n$ and $M = \text{span}\{u_1, \ldots, u_n\}$. The object of this paper is to prove that if $\{u_1, \ldots, u_n\}$ satisfies certain conditions, then for $f \in C(I, X)$ and $g \in M$ we have $\|f - g\| = \inf\{\|f - h\| : h \in M\}$ if and only if $f - g$ has at least n-zeros. An application to best local approximation in $C(I, X)$ is given.

0. Introduction

Let $I = [0, 1]$ and $C(I)$ the space of real valued continous functions. If $\{u_1, \ldots, u_n\}$ forms a T-system in $C(I)$, then the space $M = \text{span}\{u_1, \ldots, u_n\}$ is a Chebechev subspace of $C(I)$, [3 p.81]. Hence for each $f \in C(I)$ there exists a unique $g \in M$ such that

$$\|f - g\| = d(f, M) = \inf\{\|f - h\| : h \in M\}.$$

The Alternation Theorem, [3, p.75], gives an important simple characterization of $g : \|f - g\| = d(f, M)$ if and only if $f - g$ has at least n-zeros".

Chui, Shisha and Smith [4] used the Alternation Theorem to prove the existence of what they called "best local approximation" in $C(I)$.

The object of this paper is to study the Alternation Theorem and the problem of best local approximation in vector valued function spaces. It turns out

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that if we assume that the Banach space X has the so-called approximation property, and the set $\{u_1, \ldots, u_n\} \subset C(I,X)$ satisfies certain conditions, then one has an Alternation Theorem, which then applied to prove results on best local approximation in $C(I,X)$.

Throughtout this paper, if X is a Banach space, then X^* is the dual of X, and for $x \in X$, and $x^* \in X^*$ we write $\langle x, x^* \rangle$ for $x^*(x)$. The unit ball of X^* is denoted by $B_1(X^*)$. The unit mass measure at $a \in I$ is denoted by δ_a. Hence $\langle f, \delta_a \rangle = f(a)$. The set of reals is denoted by R.

1. $C(I,X)$ As Scalar Functions

For Banach spaces X and Y, let $X \overset{\vee}{\oplus} Y$ ($X \overset{\wedge}{\oplus} Y$) denote the injective (projective) tensor product of X with Y, [5. Chap 1]. It is well known that $C(I,X)$ is isometrically isomorphic to $C(I) \overset{\vee}{\oplus} X$, [5, p.9] for any Banach space X. In general it is not ture that $(X \overset{\wedge}{\oplus} Y)^* = X^* \overset{\vee}{\oplus} Y^*$. In this work, we will choose X such that $(C(I) \overset{\vee}{\oplus} X)^* = [C(I)]^* \overset{\wedge}{\oplus} X^*$. Banach spaces with the so-called approximation property satisfies such equality, [5 Chap. 1]. Further, $L^p(\mu)$, $1 \leq p \leq \infty$ and $C(I)$ have the approximation property, [2, p. 245].

Every $t \in I$ represents the unit mass measure δ_t. Hence $I \subset M(I) = C(I)^*$, the space of Borel measures on I. Further I is compact in $M(I)$ with the w^*-topology. We also have $B_1(X^*)$ is compact with the w^*-topology, by the Alaoglu Theorem. Hence $I \times B_1(X^*)$ is a compact space in the product topology.

Now every $f \in C(I,X)$ can be considered as a continuous function defined on $B_1([C(I,X)]^*) = B_1(M(I) \overset{\wedge}{\oplus} X^*)$ (Since X is assumed to have the approximation property). Since $I \times B_1(X^*)$ is closed in $[C(I,X)]^*$ and $I \times B_1(X^*) \subset B_1(M(I) \overset{\wedge}{\oplus} X^*)$, we get $I \times B_1(X^*)$ is closed in $B_1(M(I) \overset{\wedge}{\oplus} X^*)$. Here, the topology we refere to is the w^*-topology.

Finally, since for $f \in C(I,X)$ we have

$$\begin{aligned}\|f\| &= \sup_{t\in I}\|f(t)\| \\ &= \sup_{t\in I}\ \sup_{x^*\in B_1(X^*)} |\langle f(t), x^*\rangle| \\ &= \sup_{\delta_t\oplus x^*} |\langle f, \delta_t\oplus x^*\rangle|,\end{aligned}$$

we can consider, and we will, $f : I \times B_1(X^*) \to R$.

2. Vector Valued Alternation Theorem

Let $f \in C(I, X)$. We set

$$m(f) = \{\delta_t \oplus x^* : \langle f, \delta_t \oplus x^*\rangle = \|f\|, \quad t \in I,\ x^* \in B_1(X^*)\}.$$

Then one can easily prove:

Lemma 2.1. *$m(f)$ is compact in $I \times B_1(X^*)$.*
Now, let $\{u_1, \ldots, u_n\} \subset C(I, X)$. For $(t, x^*) \in I \times B_1(X^*)$ we set

$$\hat{u}(t, x^*) = (\langle u_1(t), x^* >, \cdots, < u_n(t), x^*\rangle).$$

Thus $\hat{u}(t, x^*) \in R^n$ for each $(t, x^*) \in I \times B_1(X^*)$. Then

Lemma 2.2. *Let $f \in C(I, X)$. Then the set*

$$E = \{\langle f(t), x^*\rangle\hat{u}(t, x^*) : (t, x^*) \in m(f)\}$$

is a compact set in R^n.

Proof. consider the function

$$\begin{aligned}&\psi : I \times B_1(x^*) \to R^n \\ &\psi(t, x^*) = \langle f(t), x^*\rangle\hat{u}(t, x^*).\end{aligned}$$

Since $f, u_1, \ldots, u_n$ are continuous functions, then ψ is continuous. But

$$E = \{\psi(t, x^*) : (t, x^*) \in m(f)\}.$$

But by Lemma 2.1, $m(f)$ is compact. Hence E is compact. ∎

Now, let M be subspace of $C(I,X)$ generated by $u_1,\ldots,u_n$. Hence if $g \in M$, then $g = \sum_{i=1}^{n} a_i u_i$, $a_i \in R$. Since M is finite dimensional, then for each $f \in C(I,X)$ there exists at least one $g \in M$ such that

$$\|f-g\| = d(f,M) = \inf\{\|f-h\| : h \in M\}.$$

Now we prove the Characterization Theorem [3, p.73] for the space $C(I,X)$

Theorem 2.3. *Let $f \in C(I,X)$ and $g \in M$. The following are equivalent:*
(i) $\|f-g\| = d(f,M)$
(ii) $\underline{O} = (0,\cdots,0)$ *is in the convex hull of* $E = \{(r(t,x^*)\hat{u}(t,x^*); (t,x^*) \in m(r)\}$ *in* R^n, *where* $r(t,x^*) = \langle f(t)-g(t), x^*\rangle$.

Proof. (ii)$\to$ (i). Let $r(t,x^*) = \langle f(t)-g(t), x^*\rangle$. If possible assume that g is not a best approximant to f in M. Hence there exists $h \in M$ such that $\|r-h\| < \|r\|$. Consequently,

$$\|r(t,x^*) - \langle h(t), x^*\rangle| < |r(t,x^*)| \tag{1}$$

for all $(t,x^*) \in m(r)$. Equation (1) implies that

$$r(t,x^*)\langle h(t),x^*\rangle > 0 \tag{2}$$

for all $(t,x^*) \in m(r)$.

Since $h \in M$, then $h = \sum_{i=1}^{n} b_i u_i$, for some $b_i \in R$, $i = 1,\cdots,n$. Hence

$$\langle h(t),x^*\rangle = \sum_{i=1}^{n} b_i\langle u_i(t),x^*\rangle = \langle b, \hat{u}(t,x^*)\rangle,$$

where $b = (b_1,\ldots,b_n) \in R^n$. Hence, Equation (2) implies

$$r(t,x^*)\langle b, \hat{u}(t,x^*)\rangle > 0$$

for all $(t,x^*) \in m(r)$. But $r(t,x^*) = \|r\| > 0$ for all $(t,x^*) \in m(r)$. Hence $\langle b, \hat{u}(t,x^*)\rangle > 0$ for all $(t,x^*) \in m(r)$. By Lemma 2.2, the set $E = \{r(t,x^*)\hat{u}(t, x^*) : (t,x^*) \in m(r)\}$ is compact in R^n. Hence, [3, p.19], we get

$$\underline{O} = (0,0,\cdots,0) \notin \text{Convexhull of } E.$$

Conversely. (i)→(ii). Let the vector $\underline{O} = (0,\cdots,0) \notin$ convexhull of $E = \{r(t,x^*)\hat{u}(t,x^*) : (t,x^*) \in m(r)\}$. Hence, [3, p.19], there exists $b \in R^n$ such that $\langle b,\ r(t,x^*)\hat{u}(t,x^*)\rangle > 0$ for all $(t,x^*) \in m(r)$. By Lemma 2.1, $m(r)$ is compact. Hence, there exists $\mathcal{E} > 0$ and (t_0, x_0^*) such that

$$\begin{aligned}\mathcal{E} &= \inf\{r(t,x^*)\langle b,\hat{u}(t,x^*)\rangle : (t,x^*) \in m(r)\} \\ &= r(t_0,x_0^*)\langle b,\hat{u}(t_0,x_0^*)\rangle.\end{aligned}$$

Now, let

$$K_1 = \{(t,x^*) \in I \times B_1(X^*) : r(t,x^*) < b, \hat{u}(t,x^*) \leq \mathcal{E}/2\}.$$

The set K_1 is closed (and hence compact) in $I \times B_1(X^*)$. Further $K_1 \cap m(r) = \phi$. Since r is continuous, and K_1 is compact, then $|r|$ attains its maximum on K_1 and if

$$\alpha = \max\{|r(t,x^*)| : (t,x^*) \in K_1\},$$

then $\alpha < \|r\|$. Now choose $\lambda > 0$ such that $0 < \lambda < \dfrac{\|r\| - \alpha}{\|\sum_{i=1}^{n} b_i u_i\|}$. Hence for any $(t,x^*) \in K$, we have

$$\begin{aligned}|r(t,x^*) - \lambda \sum_{i=1}^{n} b_i\langle u_1(t),x^*\rangle| &\leq |r(t,x^*)| + \lambda|\sum_{i=1}^{n} b_i\langle u_i(t),x^*\rangle| \\ &\leq \alpha + \lambda\|\sum_{i=1}^{n} b_i u_i\| \\ &< \|r\| \qquad (\text{since } \lambda < \frac{\|r\| - \alpha}{\|\sum_{i-1}^{n} b_i u_i\|}) \qquad (*)\end{aligned}$$

On the other hand, if $(t,x^*) \notin K_1$, then we can choose λ to satisfy the addifional condition $0 < \lambda < \dfrac{\mathcal{E}}{\|\sum_{i=1}^{n} b_i u_i\|^2}$. Since $r(t,x^*)\langle b, \hat{u}(t,x^*)\rangle > \mathcal{E}/2$ for all $(t,x^*) \notin K_1$, we get:

$$\begin{aligned} |r(t,x^*) - \lambda\Sigma b_i u_i(t,x^*)|^2 &\leq \|r\|^2 + \lambda\|\Sigma b_i u_i(t,x^*)\|^2 - \mathcal{E} \\ &< \|r\|^2 \qquad (\text{since } \lambda < \frac{\mathcal{E}}{\|\Sigma b_i u_i\|^2}) \qquad (**) \end{aligned}$$

Equations (*) and (**) implies that g is not a best approximant of f in M. This ends the proof. ∎

For $f \in C(I,X)$ and $x^* \in B_1(X^*)$ set:

$$m(f,x^*) = \{(t,x^*) : |\langle f(t),x^*\rangle| = \|f\|\}.$$

clearly $m(f,x^*)$ is a closed subset of $m(f)$. Then Theorem 2.3 is valid in the following setting.

Theorem 2.4. *Let $f \in C(I,X)$ and $g \in M$. The following are equivalent:*
(i) $\|f-g\| = d(f,M)$
(ii) $\underline{O} = (0,0,\cdots,0)$ *is in the convex hull of* $E(x^*) = \{r(t,x^*)\hat{u}(t,x^*) : (t,x^*) \in m(f-g,x^*)\}$.

We need one more result before we can prove the Alternation Theorem. Set

$$N = \{x^* \in B_1(X^*) : \hat{u}(t,x^*) \neq \underline{0} \text{ for all } t \in I\}.$$

For general $u_1,\ldots,u_n$ in $C(I,X)$, the set N could be empty. This occures if $u_1,\ldots,u_n$ have a common zero.

Definition 2.5. The set $\{u_1,\ldots,u_n\} \subset C(I,X)$ is said to satisfy the Haar condition if there exists at least one $x^* \in N$ such that

$$D(t_1,\ldots,t_n,x^*) = \begin{vmatrix} \langle u_1(t_1),x^*\rangle \cdots \langle u_1(t_n),x^*\rangle \\ \langle u_n(t_1),x^*\rangle \cdots \langle u_n(t_n),x^*\rangle \end{vmatrix} \neq 0$$

for all $t_1 < t_2 < \ldots < t_n$ in I. Set $N(D) = \{x^* \in N : D(t_1,\ldots,t_n,x^*) \neq 0$ for all $t_1 < t_2 < \ldots < t_n$ in $I\}$.

A simple example satisfying the Haar condition is:

$$u_1 = t \oplus x, \cdots, u_n = t^n \oplus x,$$

where $x \neq 0$ is in the Banach space X.

The set $I^n \times \{x^*\} = \{(t_1, \ldots, t_n, x^*) : t_i \in I\}$ is a convex subset of $I^n \times B_1(X^*)$. Hence the continuouity of $D(t_1, \ldots, t_n, x^*)$ as a real valued function on $I^n \times B_1(X^*)$ implies (using the Intermediate Value Theorem) that D has the same sign on $I^n \times \{x^*\}$. Then, togother with Lemma 1 [p. 74] we get:

Lemma 2.6. Let $x^* \in N(D)$ and $\{t_0, \ldots, t_n\}$ be a set of $n+1$ distinct elements in I, and $\lambda_0, \ldots, \lambda_n$ be non-zero real numbers. Let $E(x^*) = \{\lambda_0 \hat{u}(t_0, x^*), \ldots, \lambda_n \hat{u}(t_n, x^*)\}$. The $\underline{O} = (0, \cdots, 0) \in$ convexhull of $E(x^*)$ if and only if $\lambda_i \lambda_{i-1} < 0$ for $i = 1, \cdots n$.

Now we prove:

Theorem 2.7. (Alternation Theorem). *Let $\{u_1, \ldots, u_n\} \subset C(I, X)$ satisfy the Haar condition. Let $f \in C(I, X)$ and $g \in M$. The following are Equivalent:*

(i) $\|f - g\| = d(f, M)$

(ii) *$f - g$ has at least n-zeros.*

Proof. Let $r = f - g$. By Theorem 2.3, (i) is satisfied if and only if $\underline{O} = (0, \cdots, 0) \in$ convexhull of $E(x^*) = \{r(t, x^*)\hat{u}(t, x^*) : (t, x^*) \in m(r)\}$. Since $E(x^*)$ is compact, then every point in the convexhull of $E(x^*)$ is a convex linear combination of at most n-elements of $E(x^*)$. Thus there exists $\lambda_0, \ldots, \lambda_k \in (0,1)$ such that $\sum_{i=0}^{k} \lambda_i = 1$ and $0 = \sum_{i=0}^{k} \lambda_i r(t_i, x^*)\hat{u}(t_i, x^*)$. By Caratheodory Theorem [3 p.17], we have $k \leq n$.

Since the set $\{u_1, \ldots, u_n\}$ satisfy the Haar condition, the elements of any subset $\{\hat{u}(t_{i_k}, x^*) : 1 \leq k \leq n\}$ is independent in R^n. Consequently $k \geq n$. Hence $k = n$.

Lemma 2.6 now implies that $\lambda_i\, r(t_i, x^*)$ has at least $n+1$ alternation. But $\lambda_i > 0$. Thus r has at least n-zeros. This ends the proof. ∎

3. Best Local approximation

Let $\{u_1, \ldots, u_n\} \subset C(I, X)$ and $M = \text{span}\{u_1, \ldots, u_n\}$. For $\mathcal{E} \in (0,1)$ let $I_{\mathcal{E}} = [0, \mathcal{E}]$ and $M_{\mathcal{E}} = M|_{I_{\mathcal{E}}}$, the restriction of M to $I_{\mathcal{E}}$. Then $M_{\mathcal{E}} \subset C(I_{\mathcal{E}}, X)$. Let $f \in C(I, X)$ and $f_{\mathcal{E}} = f|_{I_{\mathcal{E}}}$. Since $M_{\mathcal{E}}$ is finite dimensional for all $\mathcal{E}$, it follows that for each $\mathcal{E}$ there exists $P_{\mathcal{E}}(f) \in M_{\mathcal{E}}$ such that

$$\|f_{\mathcal{E}} - P_{\mathcal{E}}(f)\| = d(f_{\mathcal{E}}, M_{\mathcal{E}}).$$

The net $(P_{\mathcal{E}}(f))$ need not to converge as $\mathcal{E} \to 0^+$. Following Chui, Shisha and Smith [4], "if $(P_{\mathcal{E}}(f))$ converges uniformly on some interval $[0, \mathcal{E}_0]$ to some $P_0(f) \in M$, then we say that $P_0(f)$ is a best local approximation of f."

The object of this section, is to use the results in section II of this paper to prove a similar type Theorems of Chui-etal [4, Theorem 2.1] for vector valued continous functions, with the uniform norm and with the L^1-norm.

For $f \in C(I, X)$ we say that f is weakly differentiable on I if for each $t \in (0,1)$

$$\lim_{\mathcal{E} \to 0} \langle \frac{f(t + \mathcal{E}) - f(t)}{\mathcal{E}}, x^* \rangle$$

exists for each $x^* \in X^*$. We will write $f'(t, x^*)$ for such limit. Let $C_\omega^n(I, X)$ denote the space of n-times weakly differnetiable functions. We let $f^{(j)}(t, x^*)$ denote the j^{th}-derevative associated with t and x^*.

Now we assume that the set $\{u_1, \ldots, u_n\} \subset C_\omega^n(I, X)$. For $x^* \in B_1(X^*)$, we let

$$A_n(x^*) = \begin{vmatrix} \langle u_1(0), x^* \rangle \cdots, \langle u_n(0), x^* \rangle \\ \vdots \qquad\qquad \vdots \\ u_1^{(n-1)}(0, x^*) \cdots u_n^{(n-1)}(0, x^*) \end{vmatrix}$$

Now we prove

Theorem 3.1. *Let $\{u_1, \ldots, u_n\} \subset C_\omega^n(I, X)$ satisfy the Haar Condition. Assume that for every $f \in C_\omega^n(I, X)$ the net $P_{\mathcal{E}}(f)$ converges uniformely to f_0 as $\mathcal{E} \to 0^+$. Then the matrix $A_n(x^*)$ is non-singular for every $x^* \neq 0$ in $B_1(X^*)$.*

Proof. Since $P_{\mathcal{E}}(f) \in M$ for each $\mathcal{E} > 0$, it follows that $P_0(f) \in M$. Further, since

$$\|P_{\mathcal{E}}(f) - f_{\mathcal{E}}\| = d(f_{\mathcal{E}}, M_{\mathcal{E}}),$$

then by Theorem 2.7 there exists $(t_i(\mathcal{E}))_{i=1}^n$, such that $0 < t_1(\mathcal{E}) < t_2(\mathcal{E}) < \cdots < t_n(\mathcal{E}) < \mathcal{E}$ and

$$P_{\mathcal{E}}(f)(t_i(\mathcal{E})) - f(t_i(\mathcal{E})) = 0, \quad (f_{\mathcal{E}}(t_i(\mathcal{E})) = f(t_i(\mathcal{E})).$$

Thus by Rolle's Theorem, for each $x^* \in$ in $B_1(X^*)$ there exists $(s_j(\mathcal{E}))_{j=1}^n$ such that

$$P_{\mathcal{E}}(f)^{j-1}(s_j(\mathcal{E}), x^*) - f^{j-1}(s_j(\mathcal{E}), x^*) = 0 \tag{1}$$

where $0 < S_1(\mathcal{E}) < \cdots < S_{n-j+1}(\mathcal{E}) < \mathcal{E}$. Now fixing x^* and taking the limit as $\mathcal{E} \to 0^+$ in (1) we get

$$\lim_{\mathcal{E}\to 0+} P_{\mathcal{E}}(f)^{j-1}(s_j(\mathcal{E}), x^*) = f^{(j-1)}(0, x^*), \tag{2}$$

$j = 1, \cdots, n$, and (2) holds for all $f \in C_\omega^n(I, X)$ and $x^* \in B_1(X^*)$.

Now, since $P_{\mathcal{E}}(f) \in M$, we have

$$P_{\mathcal{E}}(f)(t) = \sum_{i=1}^n a_i(\mathcal{E}, f) u_i(t).$$

Further, that $P_{\mathcal{E}}(f) \underset{\mathcal{E}}{\longrightarrow} P_0(f)$ and that M is finite dimensional implies that $a_i(\mathcal{E}, f) \underset{\mathcal{E}}{\longrightarrow} a_i(f)$ say for each $i = 1, \cdots, n$. Hence from Equation (2) and the fact that $u_i \in C_\omega^n(I, X)$ we get

$$\sum_{i=1}^n a_i(f) u_i^{(j-1)}(0, x^*) = f^{(j-1)}(0, x^*) \tag{3}$$

for all $x^* \in B_1(X^*)$. Since equation (3) is valid for all $f \in C_\omega^n(I, X)$, it follows that for $x^* \neq 0$, $A_n(x^*)$ is non-singular. This ends the proof. ∎

Lemma 3.2. *Let $A_n(x^*)$ be non-singular, and $\|\sum_{i=1}^n a_i(\mathcal{E}) u_i\|_{I_{\mathcal{E}}} = 0(\mathcal{E}^{n-1})$ as $\mathcal{E} \to 0^+$. Then $a_i(\mathcal{E}) \to 0$ as $\mathcal{E} \to 0^+$ for each $i = 1, \cdots, n$.*

Proof. The proof follows from the facts

(i) $\| \sum_{i=1}^{n} a_i(\mathcal{E})u_i\|_{I_\mathcal{E}} = 0(\mathcal{E}^{n-1})$ as $\mathcal{E} \to 0$ implies that $\| \sum_{i=1}^{n} a_i(\mathcal{E})\langle u_i, x^*\rangle\|_{I_\mathcal{E}} = 0(\mathcal{E}^{n-1})$ as $\mathcal{E} \to 0$

(ii) If $g(t) = \langle u_i(t), x^*\rangle$, then $g'(0) = u_i'(0, x^*)$.

(iii) Lemma 2.1 of [4]. ■

We now prove the converse of Theorem 3.1.

Theorem 3.3. *Let $A_n(x^*)$ be non-singular for each $x^* \neq 0$ in $B_1(X^*)$. Then $P_\varepsilon(f)$ coverges uniformly to some $P_0(f)$. Further, $P_0(f)^j(0, x^*) = f^j(0, x^*)$, $j = 1, 2, \cdots, n-1$ and $x^* \in B_1(X^*)$.*

Proof. Let $P_\varepsilon(f) = \sum_{i=1}^{n} a_i(\varepsilon, f)u_i$. For every $z^* \in B_1(X^*)$, the element $h = \sum_{i=1}^{n} f^{j-1}(0, x^*)u_i$ is an element of M. Since $P_\varepsilon(f)$ is the best approximant of f_ε in M_ε, it follows that

$$\|P_\varepsilon(f) - f_\varepsilon\| \leq \|h - f_\varepsilon\|_{I_\varepsilon}$$

Now, for any $g \in C(I, X)$, the map $t \to \|g(t)\|$ is continous on the compact set I. Hence $\sup_t \|g(t)\| = \|g(t_0)\|$ for some to. By the Hahn-Banach Theorem, there exists some $x^* \in B_1(X^*)$ such that

$$\sup_t \|g(t)\| = \|g(t_0)\| = \langle g(t_0), x^*\rangle$$

Consequently, since $h, f \in C(I, X)$, there exists some $t \in P_\varepsilon$ and $x^* \in B(X^*)$ such that

$$\begin{aligned} \|h - f_\varepsilon\| &= |\langle h(t) - f_\varepsilon(t), x^*\rangle| \\ &= |\sum_{i=1}^{n} f^{(i-1)}(0, x)\langle u_i(t), x^*\rangle - \langle f_\varepsilon(t), x^*\rangle| \end{aligned} \quad (1)$$

Since $A_n(x^*)$ has an inverse, we can assume that

$$\langle u_i^{(i-1)}(0), x^*\rangle = \delta_{ij} = \begin{Bmatrix} 1 & \text{if} & i = j \\ 0 & \text{if} & i \neq j \end{Bmatrix},$$

Thus expanding each $\langle u_i, x^*\rangle$ arround the point $t = 0$, using the Taylor expansion with remainder we get:

$$\begin{aligned}\langle u_i(s), x^*\rangle &= \langle u_i(0), x^*\rangle + \cdots + \frac{\langle u_i^{n-1}(0), x^*\rangle}{(n-1)!}s^{n-1} + \frac{\langle u_i^{(n)}(r), x^*\rangle}{n!}s^n \\ &= \langle u_i^{(i)}(0), x^*\rangle + R_i, \\ \text{where} \qquad R_i &= \frac{\langle u_i^{(n)}(r), x^*\rangle}{n!}s^n, \qquad 0 < r < s < \varepsilon.\end{aligned}$$

$$\text{Hence} \quad \sum_{i=1}^{n} f^{(i-1)}(0, x^*)\langle u_i(t), x^*\rangle = \sum_{i=1}^{n} f^{(i-1)}(0, x^*)\frac{t^i}{i!} + \sum_{i=1}^{n} R_i.$$

Similarly, we expand $\langle f_\varepsilon(t), x^*\rangle$ around $t = 0$, using Taylor series with remainder. Using the fact that u_i and f are in $C_\omega^n(I, X)$ for $i = 1, \cdots, n$, and the fact each R_i has the form

$$R_i = \theta(r) \cdot S^n,$$

where θ is continous and $O < r < s < \varepsilon$, it follows that for $\varepsilon \to O^+$:

$$\left|\sum_{i=1}^{n} f^{i-1}(o, x^*)\langle u_i(t), x^*\rangle - \langle f_\varepsilon(t), x^*\rangle\right| = O(\varepsilon^n) = o(\varepsilon^{n-1}) \tag{2}$$

It follows from (1) that for $\varepsilon \to 0^+$:

$$\|P_\varepsilon(f) - f_\varepsilon\| = o(\varepsilon^{n-1}) \tag{3}$$

Now, let $b_i(\varepsilon, f) = a_i(\varepsilon, f) - f^{(i-1)}(o, x^*)$. Then

$$\begin{aligned}\left\|\sum_{i=1}^{n} b_i(\varepsilon, f)u_i\right\| &= \left\|\sum_{i=1}^{n} a_i(\varepsilon, f)u_i - \sum_{i=1}^{n} f^{(i-1)}(o, x^*)u_i\right\| \\ &\le \|P_\varepsilon(f) - f_\varepsilon\| + \left\|\sum_{i=1}^{n} f^{(i-1)}(o, x^*)u_i - f_\varepsilon\right\|.\end{aligned}$$

Then using equation (2) and (3) to get for $\varepsilon \to O^+$:

$$\left\|\sum_{i=1}^{n} b_i(\varepsilon, f)u_i\right\| = o(\varepsilon^{n-1}) \tag{4}$$

Hence,

$$\lim_{\varepsilon\to 0^+} \|P_\varepsilon(f) - \sum_{i=1}^{n} f^{(i-1)}(o,x^*)u_i\|_{I_\mathcal{E}} = 0$$

Thus $P_\varepsilon(f)$ converges uniformly to some $P_0(f)$ as $\varepsilon \to O^+$. Further, equation (4) and Lemma 3.2 implies that $\lim_{\varepsilon\to 0^+} b_i(\varepsilon, f) = 0$. Hence

$$\lim_{\varepsilon\to O^+} a_i(\varepsilon, f) = f^{i-1}(o,x^*).$$

Thus

$$P_0(f)^j(o,x^*) = f^j(o,x^*).$$

This ends the proof of the Theorem.

Closing Remarks. One can consider the problem of best local approximation for different subspace M. Indeed Let $X = \ell^p$, $1 \le p < \infty$, and $L^p(I,\ell^p)$ be the space of p-Bachner integrable functions defined on I with values in ℓ^p. Hence for $f \in L^p(I,\ell^p)$,

$$\|f\|_p = (\int_0^1 \|f(t)\|^P dt)^{1/p} = (\int_0^1 \sum_{n=1}^{\infty} |f_n(t)|^P dt)^{1/p},$$

where $f(t) = (f_n(t))_{n=1}^{\infty}$.

Let $\{u_i, \ldots, u_n\}$ be continuous functions in $L^P(I,\ell^P)$ such that

$$u_j = (u_{ji})_{i=1}^{\infty},$$

and $\{u_{1i}, \ldots, u_{ni}\}$ is a T-system in $C(I)$ for each $i = 1,2,3,\cdots$. Set $M =$ span of $\{u_i, \ldots, u_n\}$. In [6] Kroo proved that M is a Chebechev subspace in $L^1(\ell_n^2)$, ℓ_n^2 is a finite dimensional Hilbert space. The authors proved in [1] that M is a Chebechev subspace in $L^1(\ell^p)$, for any $1 \le p \le \infty$, and with no restriction on the dimension of ℓ^p. If we set $p = 1$ and $M_j = \text{span}\{u_{11}, \ldots, u_{nj}\}$, then each M_j is a Chebechev subspace in $L^1(I)$ by the Jackson's Theorem [3]. Set

$$M = \{(g_j) : g_j \in M_j : \int_0^1 \|g(t)\| dt < \infty\}.$$

Then M is a closed subspace of $L^1(I,\ell^1)$. Further M is proximinal. For $f = (f_n) \in L^1(I,\ell^1)$ and $\hat{f}_n \in M_n$ such $d(f_n, M_n) = \|f_n - \hat{f}_n\|_1$, we have $g = (\hat{f}_n) \in M$ and

$$d(f,M) = \|f-g\|.$$

In this case the problem of best local approximation is that for the coordinate functions f_n, and one can prove

Theorem. *$P_\mathcal{E}(f)$ converges in $L^1(I,\ell^1)$ to some $P_0(f)$ in M if only if $P_\mathcal{E}(f_n)$ converges in $L^1(I)$ to some $P_0(f_n)$ in M_n.*

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