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STRONG F_A-SUMMABILITY

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Abstract. Let $A = (a_{nk})$ be an infinite matrix and $x = (x_k)$ an infinite sequence of complex numbers. A sequence x is said to be F_A-summable to a number ℓ [Acta Math. 80 (1948), 167-190] if and only if x is bounded and

$$\sum_{k=0}^{\infty} a_{nk} x_{k+p} \to \ell$$

as $n \to \infty$, uniformly for $p \geq 0$.

The object of this paper is to define strong F_A-summability which is a generalization of strong almost convergence due to I. J. Maddox [Math. Proc. Camb. Phil. Soc., 83 (1978), 61-64]. We also characterize the matrices which transform strong almost convergent sequences to strong F_A-summable sequences.

1. Introduction

Let ℓ_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)_0^\infty$ respectively, with $\|x\| = \sup_{k\geq 0} |x_k|$. Let D be the shift operator on the set of all real or complex sequences, i.e., $Dx = (x_k)_1^\infty$, $D^2x = (x_k)_2^\infty$ and so on. It may be recalled that [1], the Banach limit L is a non-negative linear functional on ℓ_∞ such that L is invariant under the shift operator i.e. $L(Dx) = L(x)$ for all $x \in \ell_\infty$ and that $L(e) = 1$ where $e = (1, 1, 1, \cdots)$. A sequence $x \in \ell_\infty$ is called almost convergent [2] if all of its Banach limits coincide.

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Let f denote the set of all almost convergent sequences, i.e.

$$f = \left\{ x \in \ell_\infty : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} x_{k+p} \text{ exist uniformly in } p \right\}.$$

Lorentz [2] further defined a sequence x to be F_A-summable to a number ℓ if and only if $x \in \ell_\infty$ and

$$\sum_{k=0}^{\infty} a_{nk}\, x_{k+p} \to \ell, \text{ as } n \to \infty, \text{ uniformly in } p \geq 0.$$

If $a_{nk} = \frac{1}{n}$ for $k < n$, and $= 0$ for $k \geq n$. Then F_A-summability is same as almost convergence of x. Furthermore, if we take $p = 0$, the above definition reduces to that of A-summability.

We use the notation $x_k \to \ell(F_A)$ to denote x to be F_A-summable to ℓ, and (F_A) to denote the set of all F_A-summable sequences.

The summability methods of real or complex sequences by infinite matrices are of three types: Ordinary, strong and absolute. It is therefore naturally expected here that the concept of F_A-summability must give rise to other two types, i.e., strong F_A-summability and absolute F_A-summability. In this paper we introduce strong F_A-summability and its related sublinear functional. We also characterize the matrices which transform the elements of $[f]$ to $[F_A]$. We further extend the space $[F_A]$ and study certain properties.

2. Strong F_A-Summability

In this section, we define strong analogue to F_A-summability in the following manner.

Definition 2.1. A sequence $x \in \ell_\infty$ is said to be strong F_A-summable to a number ℓ, i.e. $x_k \to \ell[F_A]$ if and only if

$$\sum_{k=0}^{\infty} a_{nk} |x_{k+p} - \ell| \to 0, \text{ as } n \to \infty, \text{ uniformly in } p \geq 0.$$

By $[F_A]$ and $[F_A]_0$ we mean the spaces of sequences which are stronly F_A-summable, and strongly F_A-summable to zero respectively.

It is easy to see that

$$|\sum_k a_{nk}(x_k - \ell)| \leq \sum_k a_{nk}|x_k - \ell|.$$

Hence $[F_A] \subset F_A$, and strongly F_A-summability is equivalent to F_A-summability if $a_{nk} = M$ (a constant) $< \infty$ for each k, and for fixed n.

Remark. If $A = (a_{nk})$ with

$$a_{nk} = \begin{cases} \frac{1}{n}, & k < n, \\ 0, & k \geq n, \end{cases}$$

then $[F_A] = [f]$, the space of strongly almost convergent sequences [4].

We define the following sublinear functionals

$$S(x) = \limsup_n \sup_p \sum_k a_{nk}\, x_{k+p}$$
$$\mathcal{L}(x) = \limsup_n \sup_p \sum_k a_{nk}|x_{k+p}|$$

Theorem 2.1. (i) $(F_A) = \{x \in \ell_\infty : S(x) = -S(-x)\}$,
(ii) $[F_A] = \{x \in \ell_\infty : \mathcal{L}(x - \ell e) = 0$ *for some* $\ell\}$.

Proof. (i) Let $x \in \ell_\infty$ and $S(x) = -S(-x)$, i.e.,

$$\limsup_n \sup_n \sum_k a_{nk}\, x_{k+p} = \liminf_n \inf \sum_k a_{nk}\, x_{k+p}$$

which holds if and only if

$$\sum_k a_{nk}\, x_{k+p}$$

tends to a limit as $n \to \infty$. Hence $x \in (F_A)$.

(ii) It is easy to see that $x \in [F_A]$ if and only if $x \in \ell_\infty$ and

$$\mathcal{L}(x - \ell e) = -\mathcal{L}(\ell e - x).$$

But $\mathcal{L}(x) = \mathcal{L}(-x)$. Therefore

$$\mathcal{L}(x - \ell e) = 0.$$

3. Matrix Transformation

The following theorem characterizes the matrices $B \in ([f],[F_A])$ i.e. $Bx \in [F_A]$ whenever $x \in [f]$ for an infinite matrix $B = (b_{nk})$, where $Bx = (\sum_k b_{nk}\, x_k)_n$ exists.

Theorem 3.1. *Let $A = (a_{nk})$ be an infinite matrix such that*

(i) $\|A\| < \infty$, *and*

(ii) $\sum_{k=0}^{\infty} a_{nk}$ *converges for each* n.

Then $B = (b_{nk}) \in ([f],[F_A])$ if and only if

(iii) $\|B\| < \infty$,

(iv) *For each $k = 0, 1, 2, \cdots$, there exists b_k such that*

$$(b_{nk} - b_k) \in [F_A]_0$$

(v) *For each set E which is uniformly of zero density (see [4]).*

$$(\sum_{k \in E} b_{nk} - b_k) \in [F_A]_0,$$

and

(vi) $\sum_{k=0}^{\infty} |b_k| < \infty$.

Proof. *Necessity.* Let $B \in ([f],[F_A])$. Condition (iii) follows by $([f],[F_A]) \subset (c, \ell_\infty)$. Necessity of (iv) is obvious. To prove the necessity of (vi), we note that

$$|b_k| \leq \sup_n |b_{nk}|.$$

Thus, for each $i = 0, 1, 2, \cdots$.

$$\sum_{k=0}^{i} |b_k| \leq \sup_n \sum_{k=0}^{i} |b_{nk}| \leq \|B\|.$$

To prove (v), let $E = \{j(0), j(1), \cdots.\}$ be an infinite set which is uniformly of zero density. Define the matrix $C = (c_{nk})$ by

$$c_{nk} = b_{n,j(k)}.$$

$C \in (\ell_\infty, [F_A])$, therefore, uniformly in p

$$\lim_m \sum_n a_{mn} | \sum_k c_{n+p,k}\, x_k - \sum_k b_k\, x_k|$$

$$= \lim_m \sum_n a_{mn} \left| \sum_{j(k)\in E} (b_{n+p,j(k)} - b_k) x_k \right|$$

$$= 0, \text{ by (i), (iii) and (iv).}$$

Hence (v) follows for $x \in [f]$.

Sufficiency. Let $x_k \to \ell[f]$. Then for each m

$$\sum_n a_{mn} \sum_k b_{n+p,k}\, x_k$$

$$= \ell \sum_n a_{mn} \sum_k b_{n+p,k} + \sum_k (x_k - \ell) \sum_n a_{mn}\, a_{n+p,k}.$$

Let

$$(\sum_k a_{mn}\, b_{n+p,k}) - b_k = \beta(m,n,p).$$

Then, by (i), (ii) and (v)

$$\sum_n |\beta(m,n,p)| \le 2\|B\|\, \|A\|$$

for all m. Therefore, by (iv) and (v)

$$\lim_m \sum_n (x_k - \ell)\, \beta(m,n,p) = 0.$$

Hence $Bx = (\sum_k b_{nk}\, x_k)_n$ is bounded and

$$\lim_m \sum_n a_{mn} | \sum_k b_{n+p,k}\, (x_k - \ell)| = 0, \text{ uniformly in } p,$$

which follows

$$\sum_k b_{nk}\, x_k \to \ell(b - \sum_k b_k) + \sum_k b_k\, x_k(F_A)$$

for $x_k \to \ell[F_A]$.

This completes the proof of the theorem.

4. Extension of $[F_A]$

Let $q = (q_k)$ be a sequence of positive real numbers. Write

$$T_{n,p}(x) \;=\; \sum_k a_{nk}|x_{k+p}|^{q_k}$$

if the series converges for each n and p.

We extend the space $[F_A]$ to

$$[F_A, q] \;=\; \{x : T_{n,p}(x - \ell e) \to 0, \text{ as } n \to \infty, \text{ uniformly in } p\}.$$

Theorem 4.1. *Suppose that* $\|A\| < \infty$, $0 < q_k \le r_k$ *and* $\dfrac{r_k}{q_k}$ *is bounded. Then*

$$[F_A, r] \subset [F_A, q]$$

Proof. Define

$$u_{k,p} \;=\; \begin{cases} y_{k,p}, & y_{k,p} \ge 1, \\ 0, & y_{k,p} < 1, \end{cases}$$

and

$$v_{k,p} \;=\; \begin{cases} 0, & y_{k,p} \ge 1, \\ y_{k,p}, & y_{k,p} < 1, \end{cases}$$

where

$$y_{k,p} \;=\; |x_{k+p} - \ell|^{r_k}.$$

Therefore

$$y_{k,p} \;=\; u_{k,p} + v_{k,p},$$

and

$$y_{k,p}^{\lambda_k} = u_{k,p}^{\lambda_k} + v_{k,p}^{\lambda_k}$$

where

$$\lambda_k = \frac{q_k}{r_k}.$$

Now, it follows that

$$u_{k,p}^{\lambda_k} \le u_{k,p} \le y_{k,p},$$

and

$$v_{k,p}^{\lambda_k} \le v_{k,p}^{\lambda} \text{ for } 0 < \lambda \le \lambda_k \le 1.$$

We have the inequality (see Maddox [3] p. 351).

$$\sum_k a_{nk}\, y_{k,p}^{\lambda_k} \le \sum_k a_{nk}\, y_{k,p} + \Big(\sum_k a_{nk}\, v_{k,p}\Big)^{\lambda} ||A||^{(1-\lambda)}$$

Hence $x \in [F_A, q]$ if $x \in [F_A, r]$.

This completes the proof of the theorem.

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References

[1] S. Banach, "Theorie des Operations Lineaires" (Warszawa 1932).

[2] G. G. Lorentz, "A contribution to the theory of divergent sequences", *Acta Math.*, 80 (1948), 167-190.

[3] I. J. Maddox, "Space of strongly summable sequences", *Quart. J. Math.* (Oxford), 18 (1967), 345-355.

[4] I. J. Maddox, "A new type of convergence," *Math. Proc. Camb. Phil. Soc.*, 83 (1978), 61-64.

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