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TOTALLY UMBILICAL SEMI-INVARIANT SUBMANIFOLDS AND CR-SUBMANIFOLDS OF A SASAKIAN MANIFOLD

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Abstract. In the present paper, a classification theorem for totally umbilical semi-invariant submanifold is established. CR-submanifolds of a Sasakian space form are studied in detail, and finally a theorem for a CR-submanifold of a Sasakian manifold to be a proper contact CR-product is proved.

1. Introduction

The notion of semi-invariant submanifold of a Sasakian manifold, which is a natural generalization of both invariant submanifolds [10] and anti-invariant submanifolds [9] in a Sasakian manifold was introduced and studied in detail by A. Bejancu and N. Papaghuic [4]. On the other hand, M. Kobayashi [12] initiated the study of CR-submanifolds of a Sasakian manifold and established that there exist no proper contact CR-product in a Sasakian space form $\overline{M}(c)$ with $C < -3$. In view of this, it was interesting to ascertain the existence of a proper contact CR-product in a Sasakian space form $\overline{M}(C)$ when $C > -3$. The purpose of the present paper is to classify semi-invariant submanifolds of a Sasakian manifold and to investigate the situation under which the CR-submanifold becomes a proper contact CR-product.

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2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, n, g) where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, n is a 1-form and g is the Riemannian metric on $\overline{M}$. These tensors satisfy [6]

$$\phi^2 x = -X + n(X)\xi, \ \phi\xi = 0, \ n(\xi) = 1, \ n(\phi X) = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - n(X)n(Y), \ n(X) = g(X, \xi)$$

for any vector fields X, Y tangent to $\overline{M}$. We denote by $\overline{\nabla}$ the covarient derivative with respect to the metric g on $\overline{M}$. It is known that $\overline{M}$ is a Sasakian manifold if and only if

$$(\overline{\nabla}_X \phi)Y = g(X,Y)\xi - n(Y)X, \ \overline{\nabla}_X \xi = -\phi X. \tag{2.2}$$

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. M is called a CR-submanifold of $\overline{M}$ if M is tangent to ξ and there exists a differentiable distribution $D : x \to D_x \subset T_xM$ such that $\phi D_x = D_x$ and $\phi D_x^{\perp} \subset T_x^{\perp} M$, where $D^{\perp}$ denotes the orthogonal complementary distribution of D and T_xM, $T_x^{\perp}M$ denote the tangent space and the normal space of M respectively. We call the pair $(D, D^{\perp})$ ξ-horizontal (resp. ξ-vertical) if $\xi \in D$ (resp. $\xi \in D^{\perp}$)[12]. M is said to be proper if neither $D = 0$ nor $D^{\perp} = 0$. For a vector field X tangent to M and N normal to M, we put

$$\phi X = PX + FX, \text{ and } \phi N = BN + CN, \tag{2.3}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of ϕX, and BN (resp. CN) denotes the tangential (resp. normal) component of ϕN. It follows that the normal bundle $T^{\perp}M$ splits as $T^{\perp}M = \phi D^{\perp} \oplus u$, where u is the orthogonal complement of $\phi D^{\perp}$ and is invariant subbundle of $T^{\perp}M$ under ϕ. Let ∇ be the Riemannian connection on M, then the Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.5}$$

for each vector fields X, Y tangent to M and N normal to M, h and A are both the second fundamental forms related by

$$g(A_N X, Y) = g(h(X,Y), N), \tag{2.6}$$

and $\nabla^{\perp}$ denotes the connection in the normal bundle $T^{\perp}M$ of M. We call the normal connection $\nabla^{\perp}$ of M to be $(D-u)$-flat if $R^{\perp}(X,Y)\,N = 0$ for $X, Y \in D$ and $N \in u$. M is called $(D-u)$ totally geodesic if $A_N\,X = 0$ for each $X \in D$ and $N \in u$.

The equation of Codazzi and Ricci are given respectively by

$$\begin{aligned}\overline{R}(X,Y,Z,N) &= g(\nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), N) \\ &\quad - g(\nabla^{\perp}_Y h(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z), N). \qquad (2.7)\\ \overline{R}(X,Y,N,N_1) &= R^{\perp}(X,Y,N,N_1) - g([A_N, A_{N_1}]X, Y) \qquad (2.8)\end{aligned}$$

for each X, Y and Z tangent to M and N, N_1 normal to M. $\overline{R}$, R and $R^{\perp}$ denote the curvature tensors associated with $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ respectively. In case of Sasakian manifold, the following equations are well known [9].

$$\begin{aligned}\overline{R}(X,Y)\phi Z &= \phi\overline{R}(X,Y)Z + g(\phi X, Z)Y - g(Y,Z)\phi X \\ &\quad + g(X,Z)\phi Y - g(\phi Y, Z)X. \qquad (2.9)\\ \overline{R}(X,Y)Z &= -\phi\overline{R}(X,Y)\phi Z + g(Y,Z)X - g(X,Z)Y \\ &\quad - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y. \qquad (2.10)\end{aligned}$$

If $\overline{M}$ is a Sasakian space form of constant ϕ-holomorphic sectional curvature C, then $\overline{R}$ is given by [6]

$$\begin{aligned}\overline{R}(X,Y)Z &= \frac{1}{4}(C+3)[g(Y,Z)X - g(X,Z)Y] \\ &\quad + \frac{1}{4}(C-1)\{n(X)n(Z)Y - n(Y)n(Z)X \\ &\quad + g(X,Z)n(Y)\xi - g(Y,Z)n(X)\xi + g(\phi Y, Z)\phi X \\ &\quad - g(\phi X, Z)\phi Y + 2g(X, \phi Y)\phi Z\}. \qquad (2.11)\end{aligned}$$

for each X, Y and Z tangent to $\overline{M}$.

The 2-form Ω on $\overline{M}$ is defined by $\Omega(X,Y) = g(X, \phi Y)$ is skew-symmetric [4], that is,

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.12}$$

and the covarient derivative of ϕ is defined by

$$\overline{\nabla}_X \phi Y = (\overline{\nabla}_X \phi)Y + \phi(\overline{\nabla}_X Y) \tag{2.13}$$

for each X, Y tangent to $\overline{M}$.

Now, let M be an m-dimensional Riemannian manifold isometrically immersed in $\overline{M}$. We assume that the structure vector field ξ on $\overline{M}$ is tangent to M and denote by $\{\xi\}$ the distribution spanned by ξ. Also we denote by TM and $T^{\perp}M$ the tangent bundle to M and respectively the normal bundle to M.

The submanifold M of the Sasakian manifold $\overline{M}$ is called semi-invariant if it is endowed with the pair of distributions $(D, D^{\perp})$ satisfying the following conditions:

(i) $T(M) = D \oplus D^{\perp} \oplus \{\xi\}$, and D, $D^{\perp}$, $\{\xi\}$ are mutually orthogonal,

(ii) the distribution D is invariant by ϕ, i.e., $\phi D_x = D_x$ for each $x \in M$,

(iii) the distribution $D^{\perp}$ is anti-invariant by ϕ, i.e. $\phi D_x^{\perp} \subset T_x^{\perp}M$ for each $x \in M$[4].

The semi-invariant submanifold M is called anti-invariant submanifold (resp. invariant submanifold) if $D = 0$ (resp. $D^{\perp} = 0$). The projection morphisms of TM to D and $D^{\perp}$ are denoted respectively by P and Q. Using this notation we have

$$X = PX + QX + n(X)\xi \tag{2.14}$$

for each X tangent to M.

The equation of Codazzi for totally umbilical semi-invariant submanifold M is given by

$$\overline{R}(X, Y, Z, N) = g(Y, Z) g(\nabla_X^{\perp} H, N) - g(X, Z) g(\nabla_Y^{\perp} H, N), \tag{2.15}$$

where X, Y, Z are vector fields on M and $N \in T^{\perp}M$, H being the mean curvature vector.

For totally umbilical semi-invariant submanifold M, the equations (2.4) and (2.5) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)H \tag{2.16}$$

$$\overline{\nabla}_X N = -g(H,N)X + \nabla_X^{\perp} N \tag{2.17}$$

A semi-invariant submanifold M of $\overline{M}$ is said to be semi-invariant product if the distribution $D \oplus \{\xi\}$ is involutive and locally M is a Riemannian product $M_1 \times M_2$ where M_1 (resp. M_2) is a leaf of $D \oplus \{\xi\}$ (resp. $D^{\perp}$) [3].

3. Totally Umbilical Semi-Invariant Submanifolds

An $m(\geq 2)$-dimensional submanifold of an arbitrary Riemannian manifold M is called an extrensic sphere if it is totally umbilical and has nonzero parallel mean curvature vector [8]. In the present section we shall prove a classification theorem for totally umbilical semi-invariant submanifold of a Sasakian manifold. In fact we prove the following:

Theorem 3.1. *Let M, $(m \geq 5)$ be a complete connected and simply connected totally umbilical semi-invariant submanifold of a Sasakian manifold M. Then*

(1) *M is a semi-invariant product, or*

(2) *M is anti-invariant submanifold, or*

(3) *M is isometric to an ordinary sphere, or*

(4) *M is homothetic to a Sasakian manifold, or*

(5) *M is a C-totally real submanifold and the f-structure C is not parallel in the normal bundle.*

The cases (4) *and* (5) *occur only when m is odd.*

Proof. We take $Z, W \in D^{\perp}$ and using (2.4), (2.5), (2.17) and (2.2) in

(2.13) we have

$$-g(H,\phi W)Z + \nabla^{\perp}_{Z}\phi W = g(Z,W)\xi + \phi(\nabla_Z W) + \phi h(Z,W). \tag{3.1}$$

Taking inner product with Z and using the fact that M is totally umbilical we obtain

$$g(H,\phi W)\|Z\|^2 = g(Z,W)\, g(H,\phi Z). \tag{3.2}$$

Interchanging Z and W in (3.2) we get

$$g(H,\phi Z)\|W\|^2 = g(Z,W)\, g(H,\phi W). \tag{3.3}$$

(3.1) together with (3.2) gives

$$g(H,\phi W) = \frac{g(Z,W)^2}{\|Z\|^2\|W\|^2} g(H,\phi W). \tag{3.4}$$

The possible solutions of (3.4) are:
(a) $H=0$, or (b) $H^{\perp}\phi W$, or (c) $Z\|W$.
Suppose condition (a) holds, i.e., $H=0$ shows that M is totally geodesic, which ensures the first part of the theorem.

Next, suppose $H \neq 0$ and $H \in u$. Then with the help of (2.13), (2.2) and (2.1) we get $\overline{\nabla}_X \phi H = \phi \overline{\nabla}_X H$ for each $X \in D$ which further implies that

$$\nabla^{\perp}_{X}\phi H = -g(H,H)\phi X + \phi\, \nabla^{\perp}_{X} H. \tag{3.5}$$

by the use of (2.17). Since M is semi-invariant, therefore by (2.1) it follows that $\nabla^{\perp}_{X}\phi H$ and $\nabla^{\perp}_{X} H$ belongs to u. Thus $\phi X = 0$, guarantees the second part of the theorem.

Finally, suppose $H \neq 0$, $H \notin u$ and $Z\|W$, i.e., $\dim D^{\perp} = 1$. Since $\dim M \geq 5$, we can choose vectors X, $Y \in D$ satisfying $g(X,Y) = g(X,\phi Y) = 0$. For each N in $T^{\perp}M$, the equations (2.9) and (2.15) implies that $\overline{R}(\phi X, Y, \phi Y, N) = g(Y,Y)g(\phi\, \nabla^{\perp}_{\phi X}\, H, N)$ and $\overline{R}(\phi X, Y, \phi Y, N) = 0$ respectively, which further implies that $\nabla^{\perp}_{X} H = 0$. Next, we take $Z \in D^{\perp}$, $N \in u$ and $N_1 \in \phi D^{\perp}$. Then a direct consequence of (2.10) and (2.15) are $\overline{R}(Z,Y,Y,N) = 0$ and $R(Z,Y,Y,N) =$

$g(Y,Y)\ g(\nabla^{\perp}_{Z}H,N)$ respectively. Hence combining both, we obtain $\nabla^{\perp}_{Z}H \in \phi D^{\perp}$. On the same lines, one can immediately have $\overline{R}(Z,Y,Y,N_1) = 0$ and $\overline{R}(Z,Y,Y,N_1) = g(Y,Y)\ g(\nabla^{\perp}_{Z}H,N_1)$, which implies that $\nabla^{\perp}_{Z}H \in u$. Thus we have proved for $Z \in D^{\perp}$, $\nabla^{\perp}_{Z}H \in \phi D^{\perp} \cap u = \{0\}$, i.e., $\nabla^{\perp}_{Z}H = 0$. Again, using (2.10) and (2.15) we have $\overline{R}(\xi,Y,Y,N) = 0$ and $\overline{R}(\xi,Y,Y,N) = g(Y,Y)\ g(\nabla^{\perp}_{\xi}H,N)$ for each $N \in T^{\perp}M$ respectively follows that $\nabla^{\perp}_{\xi}H = 0$. Hence $\nabla^{\perp}_{x}H = 0$ for all vector fields X tangent to M, i.e., M is an extrinsic sphere. Thus parts (3), (4) and (5) follow from [13]. This theorem thus gives a complete classification of totally umbilical semi-invariant submanifold of a Sasakian manifold.

4. CR-Submanifolds of a Sasakian Space Form

In this section we shall study in detail about the mixed totally geodesic CR-submanifold of a Sasakian space form with parallel horizontal distribution. We recall that the ϕ-holomorphic bisectional curvature of $\overline{M}$ is given by [11]

$$\overline{H}(X,Y) = \overline{R}(X,\phi X,\phi Y,Y)$$

We have,

Lemma 4.1. *Let M be a mixed totally geodesic CR-submanifold of a Sasakian manifold $\overline{M}$ with parallel horizontal distribution. Then for each $X \in D$ and $Z \in D^{\perp}$,*

$$\overline{H}(X,Z) = 0$$

Proof. Taking into accoount the mixed totally geodesicness of M in (2.7) to get

$$\begin{aligned}\overline{R}(X,\phi X,Z,\phi Z) = &- g(h(\nabla_X\phi X,Z),\phi Z) - g(h(\phi X,\nabla_X Z),\phi Z)\\ &+ g(h(\nabla_{\phi X}X,Z),\phi Z) + g(h(X,\nabla_{\phi X}Z),\phi Z). \quad (4.1)\end{aligned}$$

Since D is parallel, $h(\nabla_X \phi X, Z) = 0 = h(\nabla_{\phi X} X, Z)$. Using this and (2.6), equation (4.1) yields

$$\begin{aligned}\overline{R}(X, \phi X, Z, \phi Z) &= -g(A_{\phi Z}\phi X, \nabla_X Z) + g(A_{\phi Z}X, \nabla_{\phi X} Z), \text{ or} \\ \overline{R}(X, \phi X, Z, \phi Z) &= g(\nabla_X A_{\phi Z}\phi X, Z) - g(\nabla_{\phi X} A_{\phi Z}X, Z). \quad (4.2)\end{aligned}$$

Using the mixed totally geodesicness of M and the parallelness of D in (4.2), the assertion follows.

We now state the main result of this section.

Theorem 4.1. *Let M be a Sasakian space form $\overline{M}(C)$ of constant ϕ-holomorphic sectional curvature C. In order that it may admit a mixed totally geodesic CR-submanifold M with parallel horizontal distribution D, it is necessary that $C = 1$.*

Proof. From lemma (4.1), it follows that $\overline{H}(X, Z) = 0$ for each $X \in D$ and $Z \in D^{\perp}$. Using the curvature equation (2.11) of the Sasakian space form together $\overline{H}(X, Z) = 0$ and (2.12) we obtain

$$0 = -\frac{(C-1)}{2}\, g(\phi X, \phi X)\, g(\phi Z, \phi Z).$$

i.e., $(C-1)\|\phi X\|^2 \, \|\phi Z\|^2 = 0$, which gives that $C = 1$. This completes the proof of the theorem.

The following theorem which we shall prove in Sasakian setting is well known in case of Kaehler manifold.

Theorem 4.2. *Let M be a mixed foliate, and $(D, D^{\perp})$ be ξ-horizontal CR-submanifold of a Sasakian space form $\overline{M}(C)$. If the normal connection is $(D-u)$-flat, then $C \leq 1$. The equality holds good if and only if M is $(D-u)$-totally geodesic.*

Proof. Since the normal connection is $(D-u)$-flat therefore $R^{\perp}(X, Y)N = 0$ for each $X, Y \in D$ and $N \in u$. Using this in Ricci equation (2.8) we obtain

$$\begin{aligned}\overline{R}(X, Y, N, \phi N) &= -g(A_{\phi N}X, A_n Y) + g(A_{\phi N}Y, A_n X), \text{ or} \\ \overline{R}(X, Y, N, \phi N) &= -2g(A_n X, A_N \phi Y) \text{ by [12]}. \quad (4.3)\end{aligned}$$

Next, by the use of (2.11) it is easy to obtain
$\overline{R}(X,Y,N,\phi N) = \frac{1}{2}(C-1)\, g(X,\phi Y)\, g(\phi N,\phi N)$. From (2.1) it follows that

$$\overline{R}(X,Y,N,\phi N) \;=\; \frac{1}{2}(C-1)\, g(X,\phi Y)\, g(N,N). \tag{4.4}$$

Taking N as a unit vector field of the normal subbundle u, and substracting (4.3) from (4.4) to get

$$(C-1)\, g(X,\phi Y) + 4g(A_N X, A_N \phi Y) \;=\; 0. \tag{4.5}$$

We put $X = \phi Y$ and since g is a positive definite metric, therefore from (4.5) follows that $C-1 \leq 0$ or $C \leq 1$. Moreover, if M is $(D-u)$ totally geodesic, then $A_N X = 0$ for each $N \in u$ and $X \in D$ implies that $C-1=0$ or $C=1$, which completes the proof of the theorem.

5. Proper Contact CR-Product

A CR-submanifold M of a Sasakian manifold $\overline{M}$ is called a contact CR-product if it is locally a Riemannian product of a Sasakian (invariant) submanifold $M^{\top}$ and a totally real (anti-invarient) submanifold $M^{\perp}$ of M. First we prove some basic lemmas which we use subsequently.

Lemma 5.1. *Let M be a CR-submanifold of a Sasakian manifold $\overline{M}$. Then M is D-totally geodesic if and only if $A_N X \in D$ for each $X \in D$ and $N \in \overset{\perp}{\mathrm{TM}}$.*

Lemma 5.2. *Let M be a CR-submanifold of a Sasakian manifold $\overline{M}$ and $(D, D^{\perp})$ be ξ-horizontal. Then the leaf $M^{\top}$ of D is totally geodesic in M if and only if*

$$g(A_{FZ}Y, X) \;=\; n(X)n(A_{FZ}Y) \tag{5.1}$$

for each X, Y in D and Z in $D^{\perp}$.

Proof. We take X, Y in D and Z in $D^{\perp}$. Then

$$g(Z, \nabla_Y \phi X) \;=\; -g(\nabla_Y Z, \phi X). \tag{5.2}$$

We recall that for each X, Y in D and Z in $D^\perp$, we have [11]

$$g(\nabla_Y Z, X) = g(PA_{FZ}Y, X) + n(\nabla_Y Z)n(X) - n(Z)g(Y, PX). \tag{5.3}$$

By the use of (2.1), (2.2) we obtain

$$n(\nabla_Y Z) = g(\nabla_Y Z, \xi) = g(Z, \phi Y) = 0. \tag{5.4}$$

Using (5.3), (5.4) in (5.2) with $(D, D^\perp)$ is ξ-horizontal we get

$$g(Z, \nabla_Y \phi X) = -g(PA_{FZ}Y, \phi X). \tag{5.5}$$

Taking into account (2.3), (5.5) becomes
$g(Z, \nabla_Y \phi X) = -g(\phi A_{FZ}Y, \phi X)$. By (2.1) it follows that

$g(Z, \nabla_Y \phi X) = -g(A_{FZ}Y, X) + n(A_{FZ}Y)\, n(X)$, from which our assertion follows immediately.

Finally we arrive at:

Theorem 5.1. *Let M be a D-totally geodesic, but not totally geodesic CR-submanifold of a Sasakian manifold $\overline{M}$ and $(D, D^\perp)$ be ξ-horizontal. Then M is a proper contact CR-product submanifold if the leaf of $D^\perp$ is totally geodesic in M. If in addition, $\overline{M}$ is a Sasakian space form $\overline{M}(C)$, then $C > -3$.*

Proof. Since M is D-totally geodesic, therefore by lemma (5.1) $A_{FZ}Y \in D^\perp$. A direct consequence of this is

$$n(A_{FZ}Y) = 0 \tag{5.6}$$

for each Y in D and Z in $D^\perp$. Moreover,

$$g(A_{FZ}Y, X) = 0 \tag{5.7}$$

for each X in D.

Hence (5.6), (5.7) and lemma (5.2) assures that the leaf $M^\top$ of D is totally geodesic in M, and by the hypothesis, the leaf $M^\perp$ of $D^\perp$ is also totally geodesic in M. Thus M is a contact CR-product submaifold. Furthermore, using (2.1), (2.3), (2.12) and the fact that $\|h(\xi, Z)\|^2 > 0$ for each $0 \neq Z \in D^\perp$, we get

$$g(h(\xi, Z), \phi Bh(\xi, Z)) + g(h(\xi, Z), \phi Ch(\xi, Z)) < 0 \tag{5.8}$$

Next by (2.4) and (2.2) it follows that $Ch(\xi, Z) = 0$. Finally using (2.6) together with $Ch(\xi, Z) = 0$, (5.8) becomes $g(A_{\phi Bh(\xi,Z)}\xi, Z) < 0$ which shows that

$$A_{\phi Bh(\xi,Z)}\xi \neq 0. \tag{5.9}$$

On the contrary, suppose that $D^\perp = 0$. Then by (5.9), $A_N\xi \notin D^\perp$ for some $N \in \overset{\perp}{\mathrm{TM}}$ which contradicts the fact that M is D-totally geodesic (See lemma 5.1). Thus $D^\perp$ cannot be zero, that is, M is a proper contact CR-product. The last part of the theorem follows from theorem (3.5) by M. Kobayashi [11].

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