ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR DIFFERENTIAL SYSTEMS*

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Abstract. In this paper three theorems on the existence of nontrivial periodic solutions of the system

$$dx/dt = e(y)$$
$$dy/dt = -e(y)f(x) - g(x)$$

are proved, which not only generalize some known results on the existence of periodic solutions of Liénard's system (i.e. the special form for $e(y) = y$), but also relax or eliminate some traditional assumptions.

1. Introduction

Consider the nonlinear autonomous differential system

$$dx/dt = e(y)$$
$$dy/dt = -e(y)f(x) - g(x) \tag{1.1}$$

where $e(y)$, $f(x)$ and $g(x)$ are continuous real functions defined on R, and are subject to the conditions which ensure that the initial value problems of (1.1) shall have a unique solution for every initial condition.

The system (1.1) is a generalization of the following Liénard system

$$dx/dt = y$$
$$dy/dt = -yf(x) - g(x) \qquad (1.2)$$

For the system (1.2), the question of the existence of periodic solutions has been widely investigated by many authors, and many good results have been obtained (see e.g. [1] [2] for an extensive bibliography and also [3]-[5]). It is worthwhile to generalize these results to more general nonlinear system. We shall be concerned here with sufficient conditions for (1.1) to possess non-trivial periodic solutions.

In this paper we use some new techniques to deal with the existence of non-trivial periodic solutions of (1.1) and obtain some new results (see Theorems 1-3 in Section 2). If applying these new results to the system (1.2), from Remarks in Section 2, one will find that our results generalize some known results in this direction, and, in certain important aspects our conditions are less restrictive. For example, in our results, $f(x)$ is allowed to change its sign infinitely many times, in particular, our theorem 1 and theorem 3 give conditions under which the system (1.1) has at least a non-trivial periodic solution without the traditional assumptions that $G(\pm\infty) = +\infty$ and $f(x)$ is positive (or nonnegative) when $|x|$ is large. As is well known these assumptions play important roles in the proof of many known results on the existence of periodic solutions of (1.2). In Section 2, we also give several simple examples to illustrate cases in which the new criteria can be applied whereas criteria obtained in [1]-[7] fail.

For ease of exposition, throughtout the paper, we will use the following notations

$$F(x) = \int_0^x f(s)ds \qquad G(x) = \int_0^x g(s)ds$$

and we make here the following assumptions

(H1) $ye(y) > 0$ for all $y \neq 0$; $xg(x) > 0$ for all $x \neq 0$.

(H2) For $0 < |x| \ll 1$, $f(x) \leq 0$ and $\not\equiv 0$.

2. Results and Remarks

Theorem 1. *Assume (H1) and (H2) are satisfied. Moreover, assume the following conditions (H3)-(H6) also hold*

(H3) $\varlimsup\limits_{x\to+\infty} F(x) > -\infty$, $\varliminf\limits_{x\to-\infty} F(x) < +\infty$. *(where $\varlimsup$ and $\varliminf$ represent limsup and liminf respectively.)*

(H4) $\lim\limits_{x\to+\infty} (F(x) + G(x)) = +\infty$; $\lim\limits_{x\to-\infty} (-F(x) + G(x)) = +\infty$.

(H5) $\varlimsup\limits_{g\to\pm\infty} |e(y)| > 0$.

(H6) *There exist $a \geq 0$ and $q(x) \in C^1 ((-\infty, -a], R^+)$ (or $q(x) \in C^1([a, +\infty), R^-))$ such that*

$$e(q(x))[f(x) + q'(x)] + g(x) \geq 0 \quad (\leq 0)$$

for all $x \leq -a (x \geq a)$.

Then there is at least one non-trivial periodic solution for the system (1.1).

Theorem 2. *Let (H1) and (H2) hold, and the following conditions (H7)-(H9) be satisfied*

(H7) *$e(y)$ is monotone increasing, and $e(\pm\infty) = \pm\infty$.*

(H8) *There exist $a > 0$ and $b > 0$ such that $f(x) \geq 0$ for all $x \notin (-a, b)$.*

(H9) *$F(b) > F(-a)$; $G(\pm\infty) = +\infty$.*

Then there exists at least one non-trivial periodic solution of the system (1.1).

Theorem 3. *If the conditions (H1)-(H4) and (H7) are satisfied, and the following condition (H10) holds*

(H10) *There exists $a > 0$ such that $F(a) > F(-a)$, and for all $x \geq a$, $g(-x) = -g(x)$ and $f(x) + f(-x) \geq 0$.*

Then the system (1.1) has at least one non-trivial periodic solution.

Remark 1. Theorems 1-3 are still valid if we consider the system

$$dx/dt = e(y - F(x)) \qquad (2.1)$$
$$dy/dt = -g(x)$$

which is equivalent to the system (1.1). In fact the nonlinear transformation

$$T(x,y) = (x, y + F(x))$$

carries system (2.1) into system (1.1) preserving the x-coordinates.

Remark 2. (H2) is only used to guarantee that the stationary point $(0,0)$ is local repulsive, therefore, the assumption (H2) may be replaced by any set of assumptions which guarantee that the origin is repulsive. By the way, using theorem 1-3 and the change of coordinates $x^* = x$, $y^* = -y$, and $t^* = -t$, one may consider the case "For $0 < |x| \ll 1$, $f(x) \geq 0$ and $\not\equiv 0$." which implies the origin is a local sink.

Remark 3. Obviously, if $F(\pm\infty) = \pm\infty$, then (H3) holds naturally, and (H4) also holds when $xg(x) > 0$ for all $x \neq 0$.

If $G(\pm\infty) = +\infty$, then (H4) holds when (H3) is satisfied.

It is easy to see that (H8) implies (H3).

If the condition (e) of Theorem in [3] "there exist $h \geq \delta$ and $b > 0$ such that $f(x) + |g(x)| > b > 0$ if $|x| > h$." is satisfied, then (H4) holds when the condition "$xg(x) > 0$ for all $x \neq 0$" holds.

Remark 4. If $\lim\limits_{x \to -\infty} f(x) > 0$ (or $\varlimsup\limits_{x \to +\infty} f(x) > 0$) and $\varlimsup\limits_{x \to -\infty} \{|g(x)|/f(x)\} < \sup\limits_{y \in R^+} \{e(y)\}$ (resp. $\varlimsup\limits_{x \to +\infty} \{|g(x)|/f(x)\} < \sup\limits_{y \in R^-} \{-e(y)\}$), then (H6) is satisfied if $xg(x) > 0$ for all $x \neq 0$. In fact, the above hypothesis implies that there exist $a > 0$ and $K > 0$ such that $f(x) > 0$ and $e(K) \geq |g(x)|/f(x)$ (resp. $-e(-K) \geq |g(x)|/f(x)$) for all $x \leq -a$ (resp. $x \geq a$). Hence, let $q(x) = K$ (resp. $= -K$), then (H6) follows.

Remark 5. For the system (1.2), (H5) holds naturally. Therefore, Theorem 1 of [5] is a special case of our Theorem 2 for $e(y) = y$, and, by the above Remarks, one can find that our Theorem 1 generalize the result of [3] and relax some conditions imposed in [3].

In order to compare our results with those obtained in [1]-[7], let us now see several simple examples

Example 1. The system

$$dx/dt = y^{2n+1}$$

$$dy/dt = -y^{2n+1}(x^3 + 2x^2 - x - 2)e^x - x^5(e^x + e^{-x}) \qquad (2.2)$$

has at least a non-trivial periodic solution for any nonnegative integer n.

Proof. For (2.2), we have $e(y) = y^{2n+1}$, $f(x) = (x^3 + 2x^2 - x - 2)e^x$ and $g(x) = x^5(e^x + e^{-x})$. Obviously, (H1), (H2) and (H7) are satisfied, and $g(-x) = -g(x)$ for every $x \in R$.

By some simple calculuses, we have

$$F(x) = (x^3 - x^2 + x - 3)e^x + 3$$

$$G(x) = (x^5 + 20x^3 + 120x)(e^x - e^{-x}) + (-5x^4 - 60x^2 - 120)(e^x + e^{-x}) + 240$$

Thus, it is easy to see that $F(+\infty) = +\infty$ and $F(-\infty) = 3$ and $G(\pm\infty) = +\infty$. Therefore, (H3) and (H4) hold.

Again since

$$F(x) - F(-x) = (x^3 + x)(e^x + e^{-x}) + (e^{-x} - e^x)(x^2 + 3)$$

$$f(x) + f(-x) = (x^2 - 1)(x + 2)e^x + (x^2 - 1)(2 - x)e^{-x}$$

consequently

$$\lim_{x \to +\infty} (F(x) - F(-x)) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} (f(x) + f(-x)) = +\infty$$

this implies that there exists $a > 0$ such that $F(a) > F(-a)$ and $f(x) + f(-x) > 0$ for all $x \geq a$, and hence (H10) holds. It therefore follows from Theorem 3 that the assertion of Example 1 is true.

Remark 6. For the system (2.2), it should be noted that $f(x) < 0$ for $x < -2$ and $f(x) > 0$ for $x > 1$, therefore, the result of [3] cannot be invoked to produce a periodic solution of (2.2) for the special case $n = 0$. Again since $G(\pm\infty) = +\infty$ and $F(-\infty) = 3$, consequently, Theorem 3 and Corollaries 1 and 2 of [4] are also invalid for $n = 0$.

Notice also that the proof of Example 1 remains valid if y^{2n+1} is replaced by any function $e(y)$ which satisfies (H1) and (H7).

Example 2. The following systems

$$dx/dt = arctgy$$
$$dy/dt = -(arctgy)(x^2 \sin^2 x - 2x - 2) - 2x/(1+x^2)^2 \qquad (2.3)$$

and

$$dx/dt = y$$
$$dy/dt = -y(x^2 \sin^2 x - 2x - 2) - 2x/(1+x^2)^2 \qquad (2.4)$$

have non-trivial periodic solution.

Proof. For (2.3) and (2.4), it is easy to show that (H1) and (H2) are satisfied, and

$$F(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{4}x \cos 2x + \frac{1}{8}\sin 2x - x^2 - 2x$$

This implies $F(\pm\infty) = \pm\infty$, from Remark 3 it follows that (H3) and (H4) hold.

On the other hand, for (2.3) and (2.4), we have respectively $\lim_{y\to\infty} |e(y)| = \lim_{y\to\infty} |arctgy| = \pi/2$ and $\lim_{y\to\infty} |e(y)| = \lim_{y\to\infty} |y| = +\infty$, consequently, (H5) also holds.

Again since, for (2.3) and (2.4), we have $\lim_{x\to-\infty} f(x) = +\infty$, and $\overline{\lim}_{x\to-\infty} \{|g(x)| /f(x)\} = 0 < \sup_{y\in R^+} \{e(y)\}$, therefore, it follows from Remark 4 that (H6) is satisfied. Thus, according to Theroem 1, the proof of Example 2 is completed.

Remark 7. For the system (2.4), it is easy to see that $f(x)$ is oscillatory on the interval $(0, +\infty)$, and, $\overline{\lim}_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = -\infty$, so the result of Villari [3] will fail. Again since $g(x) = 2x/(1+x^2)^2 \to 0$ (as $x \to \infty$) and $G(x) = x^2/(1+x^2) \to 1$ (as $x \to \infty$), therefore, Theorems of A. B. Dragilev and A. F. Filippov (see [1] Theorems 5.1 and 5.3), Theorem 3 of Villari [5] and the results of Huang [6] and Ding [7] cannot be applied.

We also would like to point out that, according to Remark 4, y on the right-hand side of (2.4) can be replaced by any function $e(y)$ satisfying (H5) and (H1).

3. Proofs of Theorems

We first prove a lemma.

Lemma 1. *If* (H1) *and* (H2) *are satisfied, then origin is a unique singular point of* (1.1), *and is local repulsive.*

Proof. According to (H1), it is easy to see that the origin is the only singular point of (1.1), and for $0 < C_0 \ll 1$, the curve

$$V(x,y) \; = \; G(x) + \int_0^y e(s)ds = C_0$$

is a closed curve surrounding the origin. Thus, it is obvious, based on $dV/dt|_{(1.1)}$ $= -e^2(y)f(x)$ and (H2), that the origin is local repulsive, and the lemma is proved.

By Lemma 1, if one can prove that there exists a bounded positive semi-trajectory for (1.1) under the conditions of Theorem 1 or 2 or 3, then the Poincaré-Bendixson theorem can be invoked to produce a non-trivial soltion of (1.1).

(I). Proof of Theorem 1

We first consider the case $q(x) \in C^1 \left((-\infty, -a], R^+\right)$ in (H6).

Let $x_0 < -a$, $y_0 > q(x_0)$, and T_A^+ be a positive semi-trajectory of (1.1) passing through point $A(x_0, y_0)$ at time t_0, and let its parametric equation be

$$x = x(t), \quad y = y(t) \quad t \geq t_0$$

Since $ye(y) > 0$ for all $y \neq 0$, it follows, when t increases, that the point $(x(t), y(t))$ moves along T_A^+ from the left (resp. right) to the right (resp. left)

in the $y > 0$ (resp. $y < 0$) half-plane. Now we prove that T_A^+ must intersect the positive x-axis.

Suppose that T_A^+ doesn't intersect the positive x-axis for all $t \geq t_0$, then we have $y(t) > 0$ for every $t \geq t_0$, and when $t \to +\infty$, T_A^+ must belong to one of the following three possible cases

(i). $\varlimsup\limits_{t \to +\infty} x(t) < +\infty$, $\varlimsup\limits_{t \to +\infty} y(t) = +\infty$.

(ii). $\varlimsup\limits_{t \to +\infty} x(t) = +\infty$, $\varlimsup\limits_{t \to +\infty} y(t) = +\infty$.

(iii). $\varlimsup\limits_{t \to +\infty} x(t) = +\infty$, $\varlimsup\limits_{t \to +\infty} y(t) < +\infty$.

Integrating $dx/dt = e(y)$ along T_A^+ from t_0 to t, we obtain

$$x(t) \;=\; x(t_0) + \int_{t_0}^t e(y(t))dt$$

Thus, it follows from (H5) that $\varlimsup\limits_{t \to +\infty} x(t) = +\infty$ when $\varlimsup\limits_{t \to +\infty} y(t) = +\infty$, this means that the case (i) is impossible.

If the case (ii) or (iii) holds, then there exists $t_1 > t_0$ such that $x(t_1) = 0$ and $x(t) > 0$ for all $t > t_1$. Thus, $g(x)/e(y) \geq 0$ for all $t \geq t_1$, it follows that

$$y(t) \;=\; y(t_1) - \int_0^x f(s)ds - \int_0^x \frac{g(s)}{e(y(s))}ds$$
$$\leq y(t_1) - F(x(t)) \qquad (\text{as } t \geq t_1)$$

Using (H3) and the above inequality, it is not difficult to show $\varlimsup\limits_{t \to +\infty} y(t) < +\infty$, consequently, the case (ii) is also impossible.

Assume the case (iii) holds, then there exists $P > 0$ such that $0 < y(t) \leq P$ for all $t \geq t_0$.

Let $Q > \max\left\{ \max\limits_{0 \leq y \leq P} e(y), \quad 1 \right\}$, then, for all $t \geq t_1$, we have

$$y(t) \;=\; y(t_1) - \int_0^x f(s)ds - \int_0^x \frac{g(s)}{e(y(s))}ds$$
$$\leq y(t_1) - F(x) - \frac{1}{Q}G(x) \qquad (3.1)$$

Again according to (H3), there exists $k > 0$ such that $F(x) > -k$ for all $x \geq 0$. On the other hand, (H4) implies that there exists $x^* > 0$ such that $F(x) + G(x) \geq$

$Qy(t_1) + (Q-1)k$ for all $x \geq x^*$, i.e., $G(x) \geq Qy(t_1) + (Q-1)k - F(x)$ for all $x \geq x^*$. Substituting this into (3.1), we obtain

$$y(t) \leq -\frac{Q-1}{Q}(F(x)+k) < 0 \quad (\text{as } x \geq x^*)$$

This contradicts the fact that $y(t) > 0$ for all $t \geq t_0$, the contradition illustrates that the case (iii) is also impossible. Therefore, T_A^+ must intersect the positive x-axis at some point B.

By using a exactly similar fashion, we can prove that when t **increases continously**, T_A^+ must also intersect the negative x-axis at some point $C(x_C, 0)$.

If $x_C \geq -a$, by (H1) and the uniqueness of solutions for the initial value problem associated with system (1.1), it is easy to see that T_A^+ is bounded. (see Fig. 1).

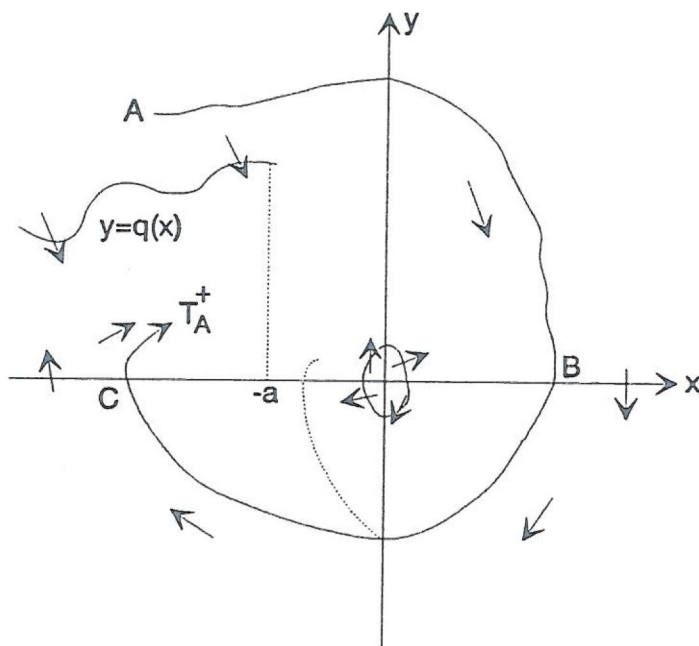

Figure 1

If $x_C < -a$, using (H6), on the curve $y = q(x)$, we have

$$\frac{-e(y)f(x)-g(x)}{e(y)} - \frac{dq(x)}{dx} \leq 0 \quad (\text{as } x \leq -a)$$

Consequently, T_A^+ cannot cross the curve $y = q(x)$ as $x \leq -a$. Arguing as in the case $x_C \geq -a$ we can know that T_A^+ is also bounded.

For the case $q(x) \in C^1([a, +\infty), R^-)$ in (H6), we choose $x_0 > a$ $y_0 < q(x_0)$, by the same argument that was used to analyze T_A^+, we can prove that the positive semi-trajectory which starting at point (x_0, y_0) must be bounded.

The Poincaré-Bendixson theorem now implies that system (1.1) has at least one non-trivial periodic solution.

(II). Proof of Theorem 2

Let

$$M = \max\{|g(x)| \mid -a \le x \le b\}$$
$$N = \max\{|f(x)| \mid -a \le x \le b\}$$

Using (H7) and (H9), we may choose $W > 0$ such that

$$F(-a) - F(b) + \frac{2M(b+a)}{\min\{e(W), |e(-W)|\}} < 0$$

Again let

$$\Delta = W + \left[N + \frac{M}{\min\{e(W), |e(-W)|\}}\right](b+a)$$
$$V(x, y) = G(x) + \int_0^y e(s)ds$$
$$C_* > \max\{V(-a, \Delta), \ V(-a, -\Delta), \ V(b, \Delta), \ V(b, -\Delta)\}$$

By (H1) and (H7) and (H9), it is clear that $V(x, y) = C_*$ is a closed curve surrounding the origin, and from our choice of C_*, it follows that $V(x, y) = C_*$ contains the region $\{(x, y) \mid -a \le x \le b, |y| \le \Delta\}$ in its interior. Thus, the curve $V(x, y) = C_*$ must intersect the lines $x = -a$ and $x = b$ in the $y > 0$ and $y < 0$ half-plane. Let their intersection points be $A(-a, y_A)$, $B(b, y_B)$, $C(b, y_C)$ and $D(-a, y_D)$ (see Fig.2), where y_A, $y_B > \Delta$, y_C, $y_D < -\Delta$. Again let $T_A^+ : y = y(x)$ be a positive semi-trajectory of (1.1) starting at A at time t_0. Using a proof similar to that used for Theorem 1, we can prove that T_A^+ cannot always stay in the region $\{(x, y) \mid -a \le x < b, y \ge W\}$ for all $t \ge t_0$. Following this, we will show that T_A^+ must intersect the line $x = b$ at a point $B * (b, y_{B*})$, and $y_{B*} \ge W$.
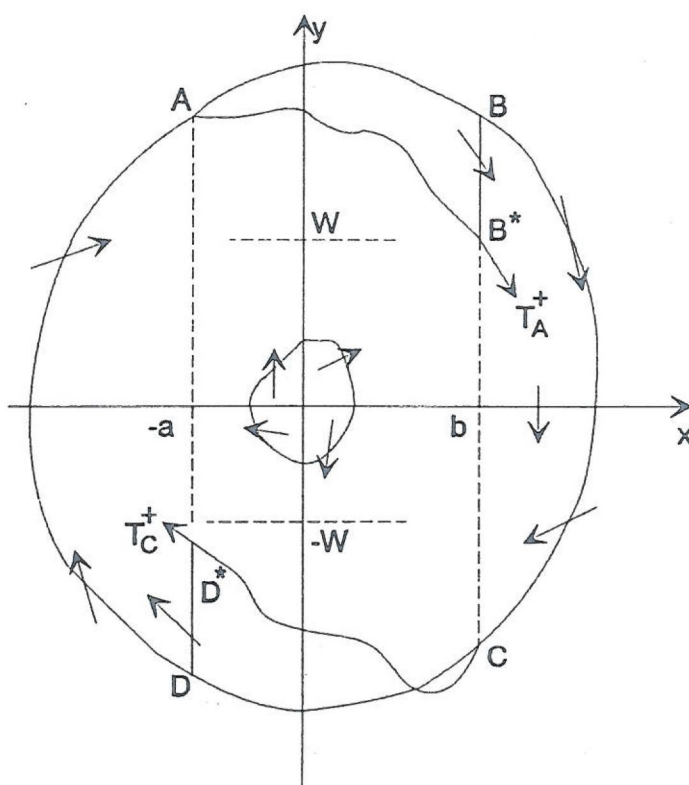
Figure 2

Otherwise, there exists $d \in (-a, b)$ such that $y(d) = W$ and $y(x) > W$ for all $x \in (-a, d)$. Again since

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{e(y)} \tag{3.2}$$

Integrating (3.2) along T_A^+ on $[-a, d]$, we obtain

$$\begin{aligned}
W - y_A &= -\int_{-a}^{d} f(x)dx - \int_{-a}^{d} \frac{g(x)}{e(y(x))}dx \\
&\geq -N(d+a) - \frac{M}{e(W)}(d+a) \geq -\left[N + \frac{M}{e(W)}\right](b+a)
\end{aligned}$$

Therefore

$$y_A \leq W + \left[N + \frac{M}{e(W)}\right](b+a) \leq \triangle$$

This contradicts $y_A > \triangle$. Hence T_A^+ must intersect the line $x = b$ at some point $B * (b, y_{B^*})$ and $y_{B^*} \geq W$.

Next we further prove $y_{B^*} < y_B$. Notice that $V(x,y) = C_*$ is a integral curve of the system

$$dx/dt = e(y) \quad dy/dt = -g(x)$$

so along this curve we have

$$y_B - y_A = -\int_{-a}^{b} \frac{g(x)}{e(y(x))} dx \geq \frac{M(b+a)}{-e(W)} \tag{3.3}$$

On the other hand, integrating (3.2) along T_A^+ from A to B^*, we obtain

$$y_{B^*} - y_A \leq F(-a) - F(b) + \frac{M(b+a)}{e(W)} < -\frac{M(b+a)}{e(W)} \tag{3.4}$$

From (3.3) and (3.4) it follows easily $y_{B^*} < y_B$.

By an analogous argument, one can prove that the positive semi-trajectory T_C^+ starting at C must intersect the line $x = -a$ at a point $D^*(-a, y_{D^*})$ in the $y < 0$ half-plane, and $y_D < y_{D^*} \leq -W$.

From (H8) it follows, along the curve $V(x,y) = C_*$, that $dV/dt\big|_{(1.1)} = -e^2(y)f(x) \leq 0$ for all $x \notin (-a,b)$. Again since $ye(y) > 0$ for every $y \neq 0$, consequently, it is easy to see that T_A^+ and T_C^+ cannot cross the closed curve $\overset{\frown}{AB^*} \cup \overset{\frown}{B^*B} \cup \overset{\frown}{BC} \cup \overset{\frown}{CD^*} \cup \overset{\frown}{D^*D} \cup \overset{\frown}{DA}$ (see Fig. 2) into its exterior. This implies that T_A^+ and T_C^+ are bounded, and the proof is completed.

(III). Proof of Theorem 3

Let

$$M^* = \max\{|g(x)|\,\big|\,|x| \leq a\} \quad N^* = \max\{|f(x)|\,\big|\,|x| \leq a\}$$

According to (H7) and (H10), we may choose $W^* > 0$ such that

$$F(-a) - F(a) + \frac{2aM^*}{\min\{e(W^*), |e(-W^*)|\}} < 0$$

Again let

$$\triangle^* = W^* + 2a\left[N^* + \frac{M^*}{\min\{e(W^*), |e(-W^*)|\}}\right]$$

In order to prove Theorem 3, we first show two lemmas.

Lemma 2. *Let $y_0 > \Delta^*$, $T^+ : x = x(t)$, $y = y(t)$ be a any positive semi-trajectory starting at the point $(-a, y_0)$ (or $(a, -y_0)$) at time t_0. Then, under the conditions of Theorem 3, there exists $t_1 > t_0$ such that T^+ intersects the line $x = a$ (resp. $x = -a$) at time t_1, and $y(t) > W^*$ (resp. $y(t) < -W^*$) for all $t \in [t_0, t_1)$, further, we also have $y(t_1) < y(t_0) = y_0$ (resp. $y(t_1) > y(t_0) = -y_0$).*

Proof. We only consider the case that T^+ starts at the point $(-a, y_0)$, since, in exactly the same way we can treat the case that T^+ starts at the point $(a, -y_0)$.

By a similar argument that was used in (II) to prove that T_A^+ intersects the line $x = b$ and $y_{B*} \geq W$, we can prove that the two first assertions of Lemma 2 are true. Now we only prove $y(t_1) < y(t_0) = y_0$.

Integrating (3.2) along T^+ from $-a$ to a, we obtain

$$\begin{aligned} y(t_1) - y_0 &= -\int_{-a}^{a} f(x)dx - \int_{-a}^{a} \frac{g(x)}{e(y(x))} dx \\ &\leq -(F(a) - F(-a)) + \frac{2aM^*}{e(W^*)} < 0 \end{aligned}$$

This implies $y(t_1) < y(t_0) = y_0$.

Lemma 3. *Let $h > a$, the trajectory $y = y(x)$ of (1.1) intersect both the lines $x = h$ and $x = -h$, and let $y(x) > 0$ (or < 0) for $x \in [-h, h]$. Then under the conditions of Theorem 3, when $y(-a) > y(a)$, we have $y(-h) > y(h)$, and the function $z(h) = y(h) - y(-h)$ is monotone nonincreasing on $h \in (a, +\infty)$.*

Proof. We only prove the case where $y(x) > 0$ for all $x \in [-h, h]$, for the case $y(x) < 0$, it can be treated in the same way.

Integrating (3.2) along the trajectory $y = y(x)$ on $[-h, h]$, we have

$$z(h) = y(h) - y(-h) = -\int_{-h}^{h} f(x)dx - \int_{-h}^{h} \frac{g(x)}{e(y(x))} dx$$

A simple calculation yields

$$\begin{aligned} \frac{d}{dh}z(h) &= -(f(h) + f(-h)) - \left[\frac{g(h)}{e(y(h))} + \frac{g(-h)}{e(y(-h))} \right] \\ &\leq \frac{g(h)[e(y(h)) - e(y(-h))]}{e(y(h))e(y(-h))} \end{aligned}$$

Now let

$$S(h) = \begin{cases} \dfrac{g(h)[e(y(h)) - e(y(-h))]}{z(h)e(y(h))e(y(-h))} & \text{when } z(h) \neq 0 \\ 0 & \text{when } z(h) = 0 \end{cases}$$

Then the above inequality means that

$$z'(h) \leq S(h)z(h) \tag{3.5}$$

where $' = d/dh$.

Using (H1) and (H7), it is not diffcult to show $S(h) \geq 0$ for $h > a$, and hence $z'(h) \leq 0$ as $z(h) = 0$. Again since $S(h)$ is continuous at h of $z(h) \neq 0$, thus, it follows from (3.5) that

$$\frac{d}{dh}\left\{ z(h)\exp\left[-\int_a^h S(\tau)d\tau \right] \right\} \leq 0$$

Therefore

$$z(h) \leq z(a)\exp\left[\int_a^h S(\tau)d\tau \right] = (y(a) - y(-a))\exp\left[\int_s^h S(\tau)d\tau \right] < 0$$

Again using (3.5), we obtain $z'(h) \leq 0$.

Thus, by what we have just proved, for any h satisfying the conditions of Lemma 3 we have $z'(h) \leq 0$. Hence $z(h)$ is monotone nonincreasing on h, it follows that $z(h) = y(h) - y(-h) \leq z(a) < 0$ for $h > a$, i.e. $y(h) < y(-h)$, and the lemma is proved.

Now let us complete the proof of Theorem 3.

Let $A_1(-a, y_{A_1})$ be a point on the straight line $x = -a$, with $y_{A_1} > \Delta^*$, and $T_{A_1}^+$ be a positive semi-trajectory of (1.1) passing through A_1. From Lemma 2 it follows that $T_{A_1}^+$ must intersect the line $x = a$ at some point A_2 in the half-plane $y > 0$. Again using a proof similar to that used for Theorem 1, we can prove that $T_{A_1}^+$ must be clockwise and intersect the positive x-axis at some point A_3, and also intersect the line $x = a$ at some point $A_4(a, y_{A_4})$ in the half-plane $y < 0$.

Choose point $B_1(a, y_{B_1})$ on the line $x = a$ such that $y_{B_1} < -\max\{\Delta^*, |y_{A_4}|\}$. Let $T_{B_1}^+$ and $T_{B_1}^-$ be respectively the positive semi-trajectory and the negative

semi-trajectory of (1.1) passing through B_1. By Lemma 2 we know that $T_{B_1}^+$ must intersect the negative y-axis at a point B_2, also intersect the line $x = -a$ in the half-plane $y < 0$. Let this intersection point be $B_3(-a, y_{B_3})$, then $y_{B_3} > y_{B_1}$. Again arguing as in (I) we may prove that $T_{B_1}^+$ must be clockwise and intersect the negative x-axis at a point B_4, and also intersect the line $x = -a$ again in the half-plane $y > 0$. Let this intersection point be $B_5(-a, y_{B_5})$.

If $y_{B_5} \leq y_{A_1}$, by the conditions, it is easy to see that $T_{B_1}^+$ cannot cross the closed curve $\overgroup{B_1 B_2 B_3 B_4 B_5} \cup \overline{B_5 A_1} \cup \overgroup{A_1 A_2 A_3 A_4} \cup \overline{A_4 B_1}$ (see Fig. 3) into its exterior. Thus, $T_{B_1}^+$ must be bounded.
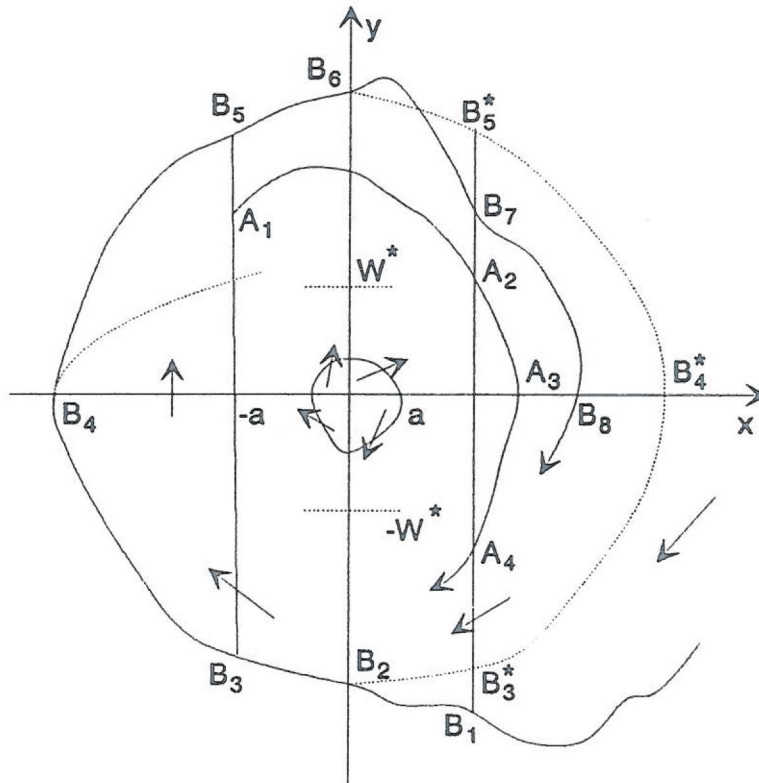

Figure 3

Now we assume $y_{B_5} > y_{A_1}$. By the same argument that was used to analyze $T_{A_1}^+$ it follows that, when t increases continuously, $T_{B_1}^+$ will intersect the positive y-axis for the first time at some point B_6, and intersect the line $x = a$ at some point $B_7(a, y_{B_7})$ in the half-plane $y > 0$, also intersect the positive x-axis at a point B_8. According to Lemma 2 we also have $W^* \leq y_{B_7} < y_{B_5}$.

We now construct a symmetrical closed curve $\overgroup{B_2 B_3 B_4 B_5 B_6} \cup \overgroup{B_6 B_5^* B_4^* B_3^* B_2}$ with respect to the y-axis (see Fig. 3), where $\overgroup{B_2 B_3 B_4 B_5 B_6}$ is a segment of $T_{B_1}^+$. By Lemma 3, it is easy to see that the segment $\overgroup{B_7 B_8}$ of $T_{B_1}^+$ will be on the left-hand side of the segmental arc $\overgroup{B_5^* B_4^*}$, and the part of $T_{B_1}^-$ which belongs to the region $\{(x,y) | x > a, y < 0\}$ must be on the right-hand side of the segmental arc $\overgroup{B_3^* B_4^*}$ (see Fig. 3). Therefore, by (H1), it is clear that $T_{B_1}^+$ is bounded.

This completes the proof of Theorem 3.

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