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EXTENSIONS GENERATED BY CLOSED SETS

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Abstract. From the nonempty collection of all closed sets (Y) of any topological space (X,τ), Schmidt generates a topological space $(Y,\mathcal{U})$. In this paper, we give some properties of this topological space. We determined when $(f,(Y,\mathcal{U}))$ is an extension of (X,τ). Also we give some separation properties. This paper leads us to unsolved problem mentioned at the end of it.

1. Introduction

Throughout the present paper X means a topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X, the complement of A will be denoted by $\sim A$. Let (X,τ) and $(Y,\mathcal{U})$ be topological spaces, and S be a dense subset of Y, if there exists a homeomorphism $f:(X,\tau)\to(S,\mathcal{U}_S)$, then the pair $(f,(Y,\mathcal{U}))$ is called an extension of (X,τ). If $(Y,\mathcal{U})$ is compact, then the extension $(f,(Y,\mathcal{U}))$ is called a compactification of (X,τ). Aspace X is called cid [3] if every countable infinite subspace of X is discrete. Aspace (X,τ) is R_1 [1] iff for $x,y\in X$ such that $\overline{\{x\}}\neq\overline{\{y\}}$ there exist disjoint open sets U and V such that $\overline{\{x\}}\subset U$ and $\overline{\{y\}}\subset V$.

Theorem 1.1. [3] (a) *A space X is cid iff every countable infinite subset is closed.*

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(b) *Any infinite cid space is* T_1.

2. General Properties

Definition 2.1. [4] Let (X, τ) be a topological space, and $Y = \tau^c - \{\phi\}$. For each open subset U of X, let $U^* = \{F \in Y : F \subset U\}$. Y with the topology generated by the collection $\beta = \{U^* : U \in \tau\}$ is denoted by $\mathcal{U}$. It is easy to prove that the collection β form a base for a topology $\mathcal{U}$ on Y. In [2] This new topology were studied especially subspaces.

Example 2.1. Let $X = \{a, b, c\}$, and $\tau = \{X, \phi, \{c\}, \{a, b\}\}$. Then $Y = \{X, \{a, b\}, \{c\}\}$ and $\mathcal{U} = \{Y, \phi, \{\{c\}\}, \{\{a, b\}\}, \{\{c\}\}, \{\{a, b\}\}\}$.

Remark 2.1. If in (X, τ) each proper open set contain no closed set then $(Y, \mathcal{U})$ is indiscrete, for example the excluded point topology [6]. The following theorem determines some properties of the elements of the collection β in any space.

Theorem 2.1. *Let* (X, τ) *be a space, then the following statments are hold for all* U *and* $V \in \tau$

(i) $(U \cap V)^* = U^* \cap V^*$,

(ii) $U^* \subseteq V^*$ *iff* $U \subseteq V$,

(iii) $U^* \neq V^*$ *iff* $U \neq V$,

(iv) $(U \cup V)^* = U^* \cup V^*$, *if* X *is cid and* T_1*-space.*

Proof. (i), (ii), and (iii) are obvious.

(iv) Let $F_1 \in (U \cap V)^* = \{F \in Y : F \subset U \cup V\}$. Then there are two cases:

(a) $F_1 \subset U$, or $F_1 \subset V$, or $F_1 \subset U \cap V$, in each case, we have $F_1 \in U^* \cup V^*$, and hence $(U \cup V)^* \subset U^* \cup V^*$. Thus $(U \cup V)^* = U^* \cup V^*$.

(b) $F_1 = F_U \cup F_V$, where $F_U \subset U$, and $F_V \subset V$. Since X is cid, we have that F_1 is countable, and F_U, F_V are countable and closed sets. Thus $F_1 \in U^* \cup V^*$, and $(U \cup V)^* = U^* \cup V^*$.

Remark 2.2. In general $(U \cup U)^* \neq U^* \cup V^*$ and we cannot drop the hypothesis "T_1" in Theorem 2.1 (iv) as the following example shows.

Example 2.2. Let $X = \{a, b, c, d, e\}$, and $\tau = \{X, \phi, \{a, b, c\}, \{b, c, d\}, \{b, c, e\}, \{b, c\}, \{a, b, c, d\}, \{a, b, c, e\}, \{b, c, d, e\}\}$. Let $U = \{b, c, d\}$ and $V = \{a, b, c\}$. Then $U^* \cup V^* = \{\{a\}, \{d\}\}$, and $(U \cup V)^* = \{\{a\}, \{d\}, \{a, d\}\}$.

3. Extension

In the following we shall answer on the question. When $(f, (Y, \mathcal{U}))$ is an extension of (X, τ).

Theorem 3.1. *Let (X, τ) be a T_1-space, then $(Y, \mathcal{U})$ is also T_1.*

Proof. Let $F_1, F_2 \in Y$, and $F_1 \neq F_2$, then there are two cases:
(a) There exists $x \in F_1$, $x \in F_1$, $x \notin F_2$, which implies that for each $y \in F_2$, $x \neq y$.
(b) There exists $y \in F_2$, $y \notin F_1$, which implies that for each $x \in F_1$, $x \neq y$. Since (X, τ) is T_1, we have for all $y \in F_2$, there exists $U_y \in \tau$ and $x \notin U_y$. This is implies that $\{x\} \not\subset U_y^*$ and $\{x\} \not\subset \bigcup\limits_{y \in F_2} U_y^* = W$, where W is open in $(Y, \mathcal{U})$.

Hence, $F_2 \in W$ and $F_1 \notin W$. Similarly $F_1 \in H$, where H is open in $(Y, \mathcal{U})$ and $F_2 \notin H$. Hence $(Y, \mathcal{U})$ is a T_1-space.

Theorem 3.2. *Let (X, τ) be a T_1-space, and we take $f : (X, \tau) \to (Y, \mathcal{U})$ such as $f(x) = \{x\}$, for each $x \in X$, then: $(f, (Y, \mathcal{U}))$ is an extension of (X, τ).*

Proof. Since X is T_1, and $Y = \tau^c - \{\phi\}$. We have that $Y = X \cup \{F \in Y : F$ is not single subset$\}$. We have the following:

(i) $X \subset Y$.

(ii) $f(X)$ is dense in $(Y, \mathcal{U})$. To prove this, Let W be any open set of $\mathcal{U}$, then $W = \cup\{U_i^* : i \in I\}$, where $U_i^* \in \beta$, and I is an arbitrary index set. Then there exists at least $U_{i_0}^* \in W$ such that $\{x\} \in U_{i_0}^*$. Thus $f(X) \cap W \neq \phi$.

(iii) The function $f : (X, \tau) \to (f(X), \mathcal{U}_{f(X)})$ is 1-1 and onto.

(iv) It remains to prove that f is a homeomorphism. Let $G \in \tau$, and $\{x\} \in f(G)$, then $x \in G$, and $\{x\} \in G^* \in W$, where $W \in \mathcal{U}$. Thus $\{x\} \in U \cap f(x)$, and $f(G) \subset U \cap f(X)$. Conversely, if $\{x\} \in U \cap f(x)$, there exists $G^* \in U \cap f(X)$ such that $x \in G \in \tau$, which implies that $\{x\} \in f(G)$. Hence $U \cap f(X) \subset f(G)$. Thus $f(G) = U \cap f(x)$. Also $f^{-1}f(G) = f^{-1}(U \cap f(x)) = G$.

4. Separation Properties

Theorem 4.1. *(X,τ) is regular iff $(Y,\mathcal{U})$ is T_2.*

Proof. First, if (X,τ) is regular, then the proof that $(Y,\mathcal{U})$ is a T_2-space is similar to the proof in Theorem 3.1. Second, let $(Y,\mathcal{U})$ is T_2, F is closed set, and $x \notin F$. We assume that $x \in F'$, where $F \neq F'$, then there exist U^* and $V^* \in \mathcal{U}$ such that $F \in U^*$ and $F' \in V^*$ such that $U^* \cap V^* = \phi$. Thus $F \subset U$ and $F' \subset V$ and $U \cap V = \phi$, $x \in V$. Hence (X,τ) is regular.

Theorem 4.2. (a) *If (X,τ) is T_4, then $(Y,\mathcal{U})$ is T_2.* (b) *If $(Y,\mathcal{U})$ is T_2, and, (X,τ) compact space, then (X,τ) is T_4.*

Proof. (a) Let $F_1, F_2 \in Y$ such that $F_1 \neq F_2$, then there exist U, $V \in \tau$ such that $F_1 \subset U$, $F_2 \subset V$, and $U \cap V = \phi$. Hence, $F_1 \in U^*$ and $F_2 \in V^*$ such that $U^* \cap V^* = (U \cap V)^* = \phi$.

(b) By using Theorem (4.1), the prove is obvious.

Theorem 4.3. *If (X,τ) is R_1, then $(Y,\mathcal{U})$ is T_4.*

Proof. It is simillar to the prove of Theorem 4.2 (a)

Unsolved Problem. Under what conditions is the last extension $(f,(Y,\mathcal{U}))$ a compactification of (X,τ)?

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