TAMKANG JOURNAL OF MATHEMATICS
Volume 24, Number 2, Summer 1993

MULTIPLIERS FOR THE $\varphi - |C, \alpha|_k$ SUMMABILITY OF INFINITE SERIES

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Abstract. In this note we improve and generalize a theorem proved in 1988 by Bor concerning $\varphi - |C, 1|_k$ summability factors of infinite series. This theorem includes several known results.

1. Introduction

Let $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}(n, k = 1, 2, \cdots)$ and let (φ_n) be a sequence of complex numbers. Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). We denote by $A_n(s)$ the A-transform of the sequence $s = (s_r)$,

$$A_n(s) = \sum_{r=1}^{\infty} a_{nr} s_r. \tag{1.1}$$

We said that the series $\sum a_n$ is summable $|A|$, if

$$\sum_{n=1}^{\infty} |A_n(s) - A_{n-1}(s)| < \infty, \tag{1.2}$$

and it is said to be summable $\varphi - |A|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} |\varphi_n[A_n(s) - A_{n-1}(s)]|^k < \infty. \tag{1.3}$$

Received April 15, 1992.

When $\varphi_n = n^{1-1/k}$ (resp. $\varphi_n = n^{\delta+1-1/k}$, $\delta \geq 0$), then $\varphi - |A|_k$ summability is the same as $|A|_k$ (resp. $|A,\delta|_k$) summability. We said that the series $\sum a_n$ is bounded $[R, \log n, 1]_k$, $k \geq 1$, if

$$\sum_{v=1}^{n} v^{-1}|s_v|^k = 0(\log n) \text{ as } n \to \infty. \tag{1.4}$$

A sequence (λ_n) is convex if $\Delta^2\lambda_n \geq 0$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, $n = 1, 2, \cdots$.

The following results are know:

Theorem A. (Mishra [4], see also Mazhar [3])

Let (λ_n) be a convex sequence such that $\sum n^{-1}\lambda_n < \infty$. If $\sum a_n$ is bounded $[R, \log n, 1]_k$, then $\sum a_n\lambda_n$ is summable $|C,1|_k$, $k \geq 1$.

Theorem B. (Balci [1])

Let (λ_n) be a convex sequence such that $\sum n^{-1}\lambda_n < \infty$. If there exists an $\in > 0$ such that the sequence $(n^{\in - k}|\varphi_n|^k)$ is non-increasing and

$$\sum_{v=1}^{n} v^{-k}|\varphi_v s_v|^k = 0\ (\log n), \quad n \to \infty, \tag{1.5}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - |C,1|_k$, $k \geq 1$.

Theorem C. (Mishra & Srivastava [5])

Let (χ_n) be a positive non-decreasing sequence and there be sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{1.6}$$

$$\beta_n \to 0 \text{ as } n \to \infty, \tag{1.7}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|\chi_n < \infty, \tag{1.8}$$

$$|\lambda_n|\chi_n = O(1). \tag{1.9}$$

If

$$\sum_{v=1}^{n} v^{-1}|s_v|^k = O(\chi_n), \tag{1.10}$$

then the series $\sum a_n\lambda_n$ *is summable* $|C,1|_k$, $k \geq 1$.

Theorem D. (Bor [2])

Let (χ_n) *be a positive non-decreasing sequence and the sequences* (β_n) *and* (λ_n) *are such that the conditions* (1.6) – (1.9) *of Theorem C are satisfied. If there exists an* $\epsilon > 0$ *such that the sequence* $(n^{\epsilon - k}|\varphi_n|^k)$ *is non-increasing and*

$$\sum_{v=1}^{n} v^{-k}|\varphi_v s_v|^k = O(\chi_n), \quad n \to \infty, \tag{1.11}$$

then the series $\sum a_n\lambda_n$ *is summable* $\varphi - |C,1|_k$, $k \geq 1$.

Theorem A could be obtained from Theorem B by taking $\varphi_n = n^{1-1/k}$ and $\epsilon = 1$. Theorem C generalizes Theorem A with conditions weaker as well. Theorems A, B and C are all special cases of Theorem D.

2. Main Result

We prove the following

Theorem E. *Let* (χ_n) *be a positive non-decreasing sequence and the sequences* (β_n) *and* (λ_n) *are such that the conditions* (1.6) – (1.9) *of Theorem C are satisfied. If there exists an* $\epsilon > 0$ *such that the sequence* $(n^{\epsilon - k}|\varphi_n|^k)$ *is non-increasing and*

$$\sum_{v=1}^{n} v^{-k\alpha}|\varphi_v t'_v|^k = O(\chi_n), \quad n \to \infty, \tag{2.1}$$

where t'_n *is the* $(C,1)$*-mean of the sequence* (na_n), *then the series* $\sum a_n\lambda_n$ *is summable* $\varphi - |C,\alpha|_k$, $k \geq 1$, $1 - 1/k < \alpha \leq 1$.

3. Lemmas

Lemma 1. [5] *If the conditions* (1.6) – (1.9) *are satisfied, then*

$$n\beta_n\chi_n = O(1), \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \beta_n \chi_n < \infty. \tag{3.2}$$

Lemma 2. *If $\sigma > \delta > 0$, then*

$$\sum_{n=v+1}^{m} \frac{(n-v)^{\delta-1}}{n^{\sigma}} = O(v^{\delta-\sigma}), \quad \textit{as } m \to \infty. \tag{3.3}$$

Proof.

$$\begin{aligned}\sum_{n=v+1}^{m} \frac{(n-v)^{\delta-1}}{n^{\sigma}} &= O(1) \int_{v+1}^{m} x^{-\sigma}(x-v)^{\delta-1} dx \\ &= O(v^{\delta-\sigma}) \int_{v/m}^{v/(v+1)} u^{\sigma-\delta-1}(1-u)^{\delta-1} du \\ &= O(v^{\delta-\sigma}),\end{aligned}$$

as

$$\begin{aligned}&\int_{v/m}^{v/(v+1)} u^{\sigma-\delta-1}(1-u)^{\delta-1} du \\ \leq &\int_{v/m}^{1} u^{\sigma-\delta-1}(1-u)^{\delta-1} du \to \beta(\sigma-\delta, \delta) \text{ as } m \to \infty.\end{aligned}$$

4. Proof of Theorem E

Let T_n^α be the n-th (C, α)-mean, $0 < \alpha \leq 1$, of the sequence $(na_n\lambda_n)$. Then in order to prove the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^k} |\varphi_n T_n^\alpha|^k < \infty, \tag{4.1}$$

where

$$\begin{aligned}T_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v, \\ A_n^\alpha &= \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Abel's transformation gives

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha}\left[\sum_{v=1}^{n-1}\{(v+1)\Delta_v A_{n-v}^{\alpha-1}\lambda_v t'_v + (v+1)A_{n-v-1}^{\alpha-1}\Delta\lambda_v t'_v\} + (n+1)\lambda_n t_n\right] \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha, \text{ say.} \end{aligned}$$

To complete the proof, it is sufficient, by Minkowski's inequality to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^k}|\varphi_n T_{n,j}^\alpha|^k < \infty, \quad j = 1, 2, 3.$$

Applying Hölder's inequality,

$$\begin{aligned} &\sum_{n=2}^{m+1} \frac{1}{n^k}|\varphi_n T_{n,1}^\alpha|^k \\ &\leq \sum_{n=2}^{m+1} \frac{|\varphi_n|^k}{n^k (A_n^\alpha)^k} \sum_{v=1}^{n-1}(v+1)^k|\lambda_v|^k|t'_v|^k|\Delta A_{n-v}^{\alpha-1}| \left\{\sum_{v=1}^{n-1}|\Delta A_{n-v}^{\alpha-1}|\right\}^{k-1} \\ &\qquad (\text{when } \alpha = 1,\ T_{n,2}^\alpha = 0 \text{ as } \Delta A_{n-v}^{\alpha-1} = 0) \\ &= O(1)\sum_{v=1}^{m} v^k|\lambda_v|^k|t'_v|^k \sum_{n=v+1}^{m+1} \frac{|\varphi_n|^k|\Delta A_{n-v}^{\alpha-1}|}{n^k (A_n^\alpha)^k}, \quad 0 < \alpha < 1 \\ &\qquad (\text{as } \sum_{v=1}^{n-1}|\Delta A_{n-v}^{\alpha-1}| = O(1)\sum_{v=1}^{n-1}(n-v)^{\alpha-2} = O(1)) \\ &= O(1)\sum_{v=1}^{m} v^k|\lambda_v|^k|t'_v|^k \sum_{n=v+1}^{m+1} \frac{|\varphi_n|^k (n-v)^{\alpha-2}}{n^{k+k\alpha}} \\ &= O(1)\sum_{v=1}^{m} v^{-k\alpha}|\varphi_v|^k\ |\lambda_v|^k\ |t'_v|^k \mathop{\Delta}_{n=v+1}^{m+1}(n-v)^{\alpha-2} \\ &= O(1)\sum_{v=1}^{m} v^{-k\alpha}|\lambda_v|^k|\varphi_v t'_v|^k \\ &= O(1)\sum_{v=1}^{m} |\lambda_v|_v^{-k\alpha}|\varphi_v t'_v|^k \\ &= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|\chi_v + O(1)|\lambda_m|\chi_m \end{aligned}$$

$$= O(1)\sum_{v=1}^{m}\beta_v\chi_v + O(1)|\lambda_m|\chi_m$$
$$= O(1),$$

in view of (1.6), (1.9), (2.1), (3.2), and the boundedness of λ_n (see [2]).

$$\sum_{n=2}^{m+1}\frac{1}{n^k}|\varphi_n T_{n,2}^{\alpha}|^k$$
$$\leq \sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n(A_n^{\alpha})^k}\sum_{v=1}^{n-1}(v+1)|\Delta\lambda_v|\,|t'_v|^k(A_{n-v-1}^{\alpha-1})^k\left\{\frac{1}{n}\sum_{v=1}^{n-1}(v+1)|\Delta\lambda_v|\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\;|t'_v|^k\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k(A_{n-v-1}^{\alpha-1})^k}{n(A_n^{\alpha})^k}$$
$$= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\;|t'_v|^k\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}|\varphi_n|^k(n-v)^{k\alpha-k}}{n^{1+\epsilon+k\alpha-k}}$$
$$= O(1)\sum_{v=1}^{m}v^{1+\epsilon-k}|\varphi_v|^k|\Delta\lambda_v|\;|t'_v|^k\sum_{n=v+1}^{m+1}\frac{(n-v)^{k\alpha-k}}{n^{1+\epsilon+k\alpha-k}}$$
$$= O(1)\sum_{v=1}^{m}v^{1-k}|\Delta\lambda_v|\;|\varphi_v t'_v|^k$$
$$= O(1)\sum_{v=1}^{m}v\beta_v v^{-k\alpha}|\varphi_v t'_v|^k$$
$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\chi_v + O(1)m\beta_m\chi_m$$
$$= O(1)\sum_{v=1}^{m}\beta_v\chi_v + \sum_{v=1}^{m}(v+1)\Delta\beta_v\chi_v + O(1)m\beta_m\chi_m$$
$$= O(1),$$

in view of (1.8), (2.1), (3.1), (3.2) and (3.3).

$$\sum_{n=1}^{m}\frac{1}{n^k}|\varphi_n T_{n,3}^{\alpha}|^k = O(1)\sum_{n=1}^{m}\frac{|\varphi_n|^k}{(A_n^{\alpha})^k}|\lambda_n|^k|t'_n|^k$$

$$= O(1) \sum_{n=1}^{m} n^{-k\alpha} |\lambda_n|^k |\varphi_n t'_n|^k$$
$$= O(1), \text{ as in the case of } T^{\alpha}_{n,1}.$$

This completes the proof of the Theorem.

5. Applications

a) If we are taking (λ_n) as a convex sequence such that $\sum n^{-1}\lambda_n < \infty$, $\chi_n = \log n$, $\in = 1$, $\varphi_n = n^{1-1/k}$ and $\alpha = 1$ in Theorem E, we obtain an improvement to Theorem A.

b) If we are taking (λ_n) as a convex sequence such that $\sum n^{-1}\lambda_n < \infty$, $\chi_n = \log n$ and $\alpha = 1$ in Theorem E, we obtain an improvement to Theorem B.

c) If we are taking $\in = 1$, $\varphi_n = n^{1-1/k}$ and $\alpha = 1$ in Theorem E, we obtain an improvement to Theorem C.

d) If we are taking $\alpha = 1$ in Theorem E, we obtain an improvement to Theorem D.

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