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AN EXTREMAL PROBLEM FOR ALGEBRAIC POLYNOMIAL

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Abstract. Some extremal properties for nonnegative polynomials of degree $\leq n$ on the interval $(-1,1)$ are proved.

Let G_n be the set of all real algebraic polynomials $P_n(x)$ of degree $\leq n$, positive on the interval $(-1,1)$. A subset of the set G_n for which $P_n^{(i-1)}(-1) = P_n^{(i-1)}(1) = 0$, $i = 1,\cdots,m$, will be denoted by $G_n^{(m)}$ $(m \leq \left[\frac{n}{2}\right])$. Let $w(x;\alpha,\beta) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$, $q = \max(\alpha,\beta)$ and $\overline{q} = \min(\alpha,\beta)$.

Let the polynomial $P_n(x)$ belongs to the set G_n. Then, for $r = 1,2,\cdots$, the polynomial $P_n(x)^r$ also belongs to the set G_n and it can be written in the form (see [1])

$$P_n(x)^r = \sum_{k=0}^{rn} C_k(1-x)^k(1+x)^{rn-k}, \quad C_k \geq 0 \ (k = 0,1,\cdots,rn). \tag{1}$$

Let

$$\|P\| = \max_{|x|\leq 1} |P(x)| \quad \text{and} \quad \|P\|_r = \left[\int_{-1}^{+1} w(x;\alpha,\beta)P_n(x)^r dx\right]^{1/r}.$$

For polynomials $P_n \in G_n$ we establish the following result:

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Theorem 1. *Let the polynomial $P_n(x)$ belongs to the set G_n and $q = \max(\alpha,\beta) \geq -\frac{1}{2}$ and $\overline{q} = \min(\alpha,\beta)$. Then*

$$\|P\| \leq \left[\frac{\Gamma(rn+\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(q+1)\Gamma(\overline{q}+rn+1)}\right]^{\frac{1}{r}} \|P\|_r. \tag{2}$$

The extremal polynomial has the form

$$P_n(x) = \begin{cases} d(1-x)^n, & q=\beta \\ d(1+x)^n, & q=\alpha, \end{cases}$$

where d is an arbitrary constant.

Proof. Let

$$m_k = \max_{|x|\leq 1}\{(1-x)^k(1+x)^{rn-k}\} = \frac{2^{rn}k^k(rn-k)^{rn-k}}{(rn)^{rn}} \qquad (m_0 = m_{rn} = 2^{rn}) \tag{3}$$

for $1 \leq k \leq rn-1$ and

$$\begin{aligned} h_k(\alpha,\beta) &= \int_{-1}^{+1} w(x;\alpha,\beta)(1-x)^k(1+x)^{rn-k}dx \\ &= \frac{2^{rn+\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(rn-k+\beta+1)}{\Gamma(rn+\alpha+\beta+2)}, \end{aligned} \tag{4}$$

for $k = 0,1,\cdots,rn$. According to (1) we obtain that

$$P_n(x)^r \leq \sum_{k=0}^{rn} c_k m_k = \sum_{k=0}^{rn} c_k h_k(\alpha,\beta)\frac{m_k}{h_k(\alpha,\beta)}. \tag{5}$$

If we assume

$$a_k^{(\alpha,\beta)} = \frac{m_k}{h_k(\alpha,\beta)} \tag{6}$$

and

$$A_n^{(0)}(\alpha,\beta) = \max_{0\leq k\leq rn}\left\{a_k^{(\alpha,\beta)}\right\}, \tag{7}$$

than according to (5) the following inequality

$$P_n(x) \leq \left[A_n^{(0)}(\alpha,\beta)\right]^{\frac{1}{r}} \|P\|_r \tag{8}$$

is obtained. To prove the inequality (2) it is necessary to determine the value $A_n^{(0)}(\alpha,\beta)$. First, let us prove that

$$A_n^{(0)}(\alpha,\alpha) = \frac{\Gamma(rn+2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1)\Gamma(rn+\alpha+1)} \tag{9}$$

for $\alpha \geq -\dfrac{1}{2}$.

Let

$$b_k(\alpha,\alpha) = \frac{a_{k+1}^{(\alpha,\alpha)}}{a_k^{(\alpha,\alpha)}} = \frac{(k+1)(rn-k+\alpha)\left[1+\dfrac{1}{k}\right]^k}{(rn-k)(k+\alpha+1)\left[1+\dfrac{1}{rn-k-1}\right]^{rn-k-1}} \tag{10}$$

$$b_0(\alpha,\alpha) = \frac{(rn+\alpha)}{(rn)(\alpha+1)\left[1+\dfrac{1}{rn-1}\right]^{rn-1}}$$

for $k = 1,\cdots,rn-1$. Then $b_k(\alpha,\alpha)$ could be written in the form

$$b_k(\alpha,\alpha) = \frac{f(k)}{f(rn-k-1)}, \quad k = 0,1,\cdots,rn-1, \tag{11}$$

where

$$f(x) = \begin{cases} \dfrac{x+1}{x+\alpha+1}\left[1+\dfrac{1}{x}\right]^x, & x > 0 \\ \dfrac{1}{1+\alpha}, & x = 0. \end{cases}$$

As $g(x) = \dfrac{f'(x)}{f(x)} = \log\left[1+\dfrac{1}{x}\right] - \dfrac{1}{x+\alpha+1}$ and $g'(x) = -\dfrac{x(2\alpha+1)+(\alpha+1)^2}{x(x+1)(x+\alpha+1)^2}$, we conclude that $g'(x) < 0$ for $\alpha \geq -\dfrac{1}{2}$ and $x > 0$. Hence, the function $g(x)$ is decreasing and $g(x) > \lim\limits_{x\to+\infty} g(x) = 0$, which proves $f'(x) > 0$ $(x > 0)$, i.e.

the function $f(x)$ is increasing for $x > 0$. Now, on the basis of (11) and the continuity of $f(x)$ at $x = 0$, we have

$$b_k(\alpha,\alpha) < 1, \text{ for } k = 0, 1, \cdots \left[\frac{rn-1}{2}\right] \tag{12}$$

and

$$b_k(\alpha,\alpha) > 1, \text{ for } k = \left[\frac{rn+1}{2}\right], \cdots, rn-1 \tag{13}$$

and further on the basis of (10)

$$a_0^{(\alpha,\alpha)} > a_1^{(\alpha,\alpha)} > \ldots > a_{[j]}^{(\alpha,\alpha)} < a_{[j+1]}^{(\alpha,\alpha)} < \ldots < a_{rn}^{(\alpha,\alpha)},$$

where $j = \dfrac{rn+1}{2}$. As $a_0 = a_{rn}$, from this inequality we obtain

$$A_n^{(0)}(\alpha,\alpha) = \max\{a_k^{(\alpha,\alpha)}\} = a_0^{(\alpha,\alpha)} = \frac{\Gamma(rn+2\alpha+2)}{2^{2\alpha+1}\Gamma(rn+\alpha+1)\Gamma(\alpha+1)},$$

which is to be proved.

Let us suppose that $q = \max\{\alpha,\beta\} = \alpha \geq -\dfrac{1}{2}$ and

$$b_k(\alpha,\beta) = \frac{a_{k+1}^{(\alpha,\beta)}}{a_k^{(\alpha,\beta)}} = \frac{(k+1)(rn-k+\alpha)\left[1+\dfrac{1}{k}\right]^k}{(rn-k)(k+1+\alpha)\left[1+\dfrac{1}{rn-k-1}\right]^{rn-k-1}},$$

where $a_k^{(\alpha,\beta)}$ is defined by (6). For such defined $b_k(\alpha,\beta)$ the inequality

$$b_k(\alpha,\beta) \leq b_k(\alpha,\alpha)$$

is valid. According to this inequality and inequality (12) we obtain

$$a_0^{(\alpha,\beta)} \geq a_1^{(\alpha,\beta)} \geq \ldots \geq a_{[j]}^{(\alpha,\beta)}, \tag{14}$$

where is $j = \dfrac{rn}{2}$. On the other hand, under condition $q = \alpha \geq -\dfrac{1}{2}$, for $k = 0, 1, \cdots, \left[\dfrac{rn}{2}\right]$, the inequality

$$\frac{a_k^{(\alpha,\beta)}}{a_{rn-k}^{(\alpha,\beta)}} = \frac{\Gamma(rn-k+1+\alpha)\Gamma(k+1+\beta)}{\Gamma(k+1+\alpha)\Gamma(rn-k+1+\beta)} = \frac{(rn-k+\alpha)\cdots(k+1+\alpha)}{(rn-k+\beta)\cdots(k+1+\beta)} \geq 1 \tag{15}$$

hold. On the basis of the (14) and (15) we obtain

$$A_n^{(0)}(\alpha,\beta) \;=\; a_0^{(\alpha,\beta)} \quad (q=\alpha \geq -\frac{1}{2}).$$

Similarly under assummption $q=\beta \geq -\frac{1}{2}$, we have

$$A_n^{(0)}(\alpha,\beta) \;=\; a_{rn}^{(\alpha,\beta)}$$

which completes the proof of the Theorem 1.

Remark. For $\alpha=\beta=0$ and $r=1$ according to inequality (8) the inequality proved in papers [2] and [3] (see also [4]) is obtain. Besides, a special case of the inequality (2), namely for $r=2$, is a result closely related to the inequaltiy proved in [5]. Similarly as in Theorem 1, the following result can be proved.

Theorem 2. *Let a polynomial $P_n(x)$ belongs to the set $G_n^{(m)}$, $m=1,\cdots,\left[\frac{n}{2}\right]$, $q=\max\{\alpha,\beta\}\geq -\frac{1}{2}$, $\overline{q}=\min\{\alpha,\beta\}$ and r is natural number. Then the inequality*

$$\|P\| \;\leq\; \left[A_n^{(m)}(\alpha,\beta)\right]^{\frac{1}{r}} \|P\|_r \tag{16}$$

holds, where

$$A_n^{(m)}(\alpha,\beta) \;=\; \frac{\left(\dfrac{m}{n-m}\right)^{rm}\left(\dfrac{n-m}{n}\right)^{rn}\Gamma(rn+\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(rm+1+q)\Gamma(rn-rm+1+\overline{q})}.$$

The equality in (16) holds for

$$P_n(x) \;=\; \begin{cases} d(1-x)^m(1+x)^{n-m}, & q=\alpha \\ d(1-x)^{n-m}(1+x)^m, & q=\beta, \end{cases}$$

where d is an arbitrary constant.

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