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RINGS WITH (R,R,R) AND $((R,R,R),R)$ IN THE LEFT NUCLEUS

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Abstract. Let R be a nonassociative ring, and N the left nucleus. It is shown that if R is a simple ring satisfying $(R,R,R) \subset N$ and $((R,R,R),R) \subset N$ and char $R \neq 2$, then R is associative.

1. Introduction

Let R be a nonassociative ring. We adopt the usual notation for associators and commutators: $(x,y,z) = (xy)z - x(yz)$ and $(x,y) = xy - yx$. We shall denote the left nucleus and the nucleus by N and G respectively. Thus N and G consists of all elements n such that $(n,R,R) = 0$ and $(n,R,R) = (R,n,R) = (R,R,n) = 0$ respectively. A ring R is called semiprime if the only ideal of R which squares to zero is the zero ideal. R is called simple if R is the only nonzero ideal of R. In [1], Kleinfeld proved that if R is a semiprime ring satisfying $(R,R,R) \subset G$ and the Abelian group $(R,+)$ has no elements of order 2, then R is associative. In [9], we improved his result as follows: If R is a semiprime ring satisfying $(R,R,R) \subset N$ and the Abelian group $(R,+)$ has no elements of order 2 and $(R,R,(R,R,R)) = 0$ then R is associative. In [8], using Kleinfeld's result [1] we proved that if R is a semiprime ring which satisfies $(R,R,R) \subset N$ and $(N,R) \subset N$ and the Abelian group $(R,+)$ has no elements of order 2, then R is associative. In this note, we weaken the second hypothesis of the last result to obtain the same result for the simple ring case. Rings with associators in the

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commutative center were studied by Kleinfeld [2] and Thedy [5]. They worked for the simple ring cases with some restrictions of characteristics. For the related results, see [3], [4], [6], [7], [10] and [11]. Note that the associator (x, y, z) and the commutator (x, y) are linear in each argument. Thus N is an additive subgroup of the additive group of R.

2. Result

Let R be a nonassociative ring. In every ring one may verify the Teichmüller identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) \ = \ w(x, y, z) + (w, x, y)z. \tag{1}$$

Suppose that $n \in N$. Then with $w = n$ in (1) we obtain

$$(nx, y, z) = n(x, y, z) \text{ for all } n \text{ in } N. \tag{2}$$

As a consequence of (2), we have that N is an associative subring of R.

We assume that R satisfies

$$(*) \quad (R, R, R) \subset N \text{ and } \quad (**) \quad ((R, R, R), R) \subset N.$$

Using (1) and (*) we have

$$w(x, y, z) + (w, x, y)z \in N. \tag{3}$$

Then with $x \in N$ in (3), we get $(R, N, R)R \subset N$. Applying this, (*) and (2), we obtain

$$(R, N, R)(R, R, R) \ = \ 0. \tag{4}$$

Then with $z \in N$ in (3), and by (*) and noting that N is an associative subring of R, we have $R(R, R, N) \subset N$. Using this and (**), we get $(R, R, N)R \subset N$. Applying this, (*) and (2), we obtain

$$(R, R, N)(R, R, R) \ = \ 0. \tag{5}$$

Definition. Let I be the associator ideal of R. I consists of the smallest ideal which contains all associators.

Note that I may be characterized as all finite sums of associators and right (or left) multiples of associators, as a consequence of (1). Hence we have

$$I = (R,R,R)+(R,R,R)R = (R,R,R)+R(R,R,R). \tag{6}$$

Using (*), (4) and (5), we have $(R,N,R)((R,R,R)R)=0$ and $(R,R,N)((R,R,R)R)=0$. Applying (6); these equalities, (4) and (5) imply

$$(R,N,R)\cdot I = 0 \text{ and } (R,R,N)\cdot I = 0. \tag{7}$$

Using Kleinfeld's result [1], we obtain the

Theorem. *If R is a simple ring satisfying $(R,R,R)\subset N$ and $((R,R,R),R)\subset N$, and char $R\neq 2$, then R is associative.*

Proof. It is easy to see that R^2 is an ideal of R. By the simplicity of R, either $R^2=0$ or $R^2=R$. If $R^2=0$ or $I=0$, then R is associative. Assume that $R^2=R$ and $I=R$. Hence, R is a semiprime ring. By (6), we have

$$R = (R,R,R)+(R,R,R)R = (R,R,R)+R(R,R,R). \tag{8}$$

Let $A=\{a\in R : aR=0\}$. Then $A\subset N$. Assume that $a\in A$, w, x, y, z, $u\in R$. Then by (**), we get

$$(x,y,z)a = ((x,y,z),a)\in N. \tag{9}$$

Since $aR=0$, by (*) we have

$$(wa)x = (w,a,x)\in N. \tag{10}$$

Using (*), (**) and (10), we obtain $((x,y,z)w)a = (x,y,z)(wa) = ((x,y,z),wa)+(wa)(x,y,z)\in N$. Applying this, (8) and (9), we have

$$Ra\subset N. \tag{11}$$

Using (11), (2) and (10), we get $(wa)(x,y,z) = ((wa)x,y,z) = 0$. Thus by this and (11), we have $(wa)((x,y,z)u) = ((wa)(x,y,z))u = 0$. By (8), the last two equalities imply $(wa)R = 0$. Hence, $Ra \subset A$. Obviously, $0 = aR \subset A$.

At this point, we have verified that A is an ideal of R and $A \cdot R = 0$. So by the simplicity of R and $R^2 = R$, we conclude that $A = 0$. By (6), (7) and (8), we get $(R,N,R) \subset A$ and $(R,R,N) \subset A$. Thus, $(R,N,R) = 0$ and $(R,R,N) = 0$. Hence $G = N$, and using (*) we have $(R,R,R) \subset G$. By Kleinfeld's result [1], $I = 0$. This contradicts to $I = R$. Thus, $I = 0$ and R is associative. This completes the proof of the theorem.

Added in proof. Recently, the author proved that if R is a semiprime ring such that the Abelian group $(R,+)$ has no elements of order 2; and satisfies $(R,R,R) \subseteq N \cap M$ and $((R,R,R),R) \subseteq N$, or satisfies $(R,R,R) \subseteq N$ and $((R,R,R),R) = 0$ then R is associative. These results are contained in an article which has been submitted for publication in this Journal entitled "Rings with associators in the left nucleus and in the middle nucleus or commutative center".

Let M and L denote the middle nucleus and right nucleus respectively. Thus M and L consists of all elements n such that $(R,n,R) = 0$ and $(R,R,n) = 0$ respectively. R is called prime if the product of any two nonzero ideals of R is nonzero. Recently, Tae-Il Suh [Prime nonassociative rings with a special derivation, Abstracts of papers presented to the Amer. Math. Soc. 14 (1993), 284] proved that if R is a prime ring with a derivation d such that $d(R) \subseteq G$ then either R is associative or $d^3 = 0$. We improved this result by concluding that either R is associative or $d^2 = 2d = 0$ under the weaker hypotheses $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$. We also that if R is a simple ring with a derivation d such that $d(R) \subseteq N \cap L$ then either R is associative or $d^2 = 2d = 0$. These results are contained in an article which has been submitted for publication in this Journal entitled "Rings with a derivation whose image is contained in the nuclei".

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