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A NOTE ON THE CONSTRUCTION OF LARGE SET OF LATIN SQUARES WITH ONE ENTRY IN COMMON

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Abstract. A latin square of order n is an $n \times n$ array such that each of the integers $1, 2, 3, \cdots, n$ occurs exactly once in each row and each column. A large set of latin squares of order n having only one entry in common is a maximum set of latin squares of order n such that each pair of them contains exactly one fixed entry in common. In this paper, we prove that a large set of latin squares of order n having only one entry in common has $n-1$ latin squares for each positive integer n, $n \geq 4$.

1. Introduction and definitions

A latin square of order n is an $n \times n$ array such that each of the integers $1, 2, 3, \cdots, n$ (or any set of n distinct symbols) occurs exactly once in each row and each column. A latin square $L = [l_{i,j}]$ is said to be *idempotent* provided that $l_{i,i} = i$. Two latin squares, $L = [l_{i,j}]$ and $M = [m_{i,j}]$ are said to *have k entries in common* if there are exactly k cells (i,j) such that $l_{i,j} = m_{i,j}$. In [2], it was shown by H. L. Fu that there exists a pair of latin squares of order n which have k entries in common for each $n \geq 5$ and k in $\{0, 1, 2, \cdots, n^2-7, n^2-6, n^2-4, n^2\}$. A large set of idempotend latin squares of order n is a set containing maximum number of idempotent latin squares of order n such that any two of them have n entries in common and these n entries are in the main diagonal. In [5], L. Terlinck and C. C. Lindner showed that a large set of idempotent latin squares

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of order n contains $n-2$ idempotent latin squares. We can extend this idea to the intersections of latin squares which are not necessarily idempotent. A large set of latin squares of order n with k entries in common is a maximum set of latin squares of order n such that each pair of them contains exactly k fixed entries in common. Let $D(n,k)$ be the cardinality of a large set of latin squares of order n with k entries in common. In [3], Fu showed that

$$\begin{aligned} D(n,n^2-4) &= 2, \text{ for } n \geq 4, \\ D(n,n^2-6) &= D(n,n^2-7) = D(n,n^2-k) = 2, \\ &\qquad \text{for } n \geq 6,\ k \in \{n^2-h \mid h = 8,10,11,13,14\}, \\ D(n,n^2-9) &= 3, \text{ for } n \geq 6 \text{ and} \\ D(n,n^2-12) &= D(n,n^2-15) = 3, \text{ for } n \geq 8. \end{aligned}$$

Then in [4], Hwang showed that $D(n,1) = n-1$ for each $n \equiv 0$ or $2 \pmod 6$. In this paper, by using a different technique, we prove that a large set of latin squares of order n with one entry in common contains exactly $n-1$ latin squares, i.e. $D(n,1) = n-1$, for each integer n, $n \geq 4$.

2. Main Construction

In [2], Fu showed the following theorem:

Theorem 2.1. *There exist a pair of latin squares of order v which have exactly k entries in common, for each*

(1) $k \in \{0,4\}$ *when* $v = 2$;

(2) $k \in \{0,3,9\}$ *when* $v = 3$;

(3) $k \in \{0,1,2,3,4,6,8,9,12,16\}$ *when* $v = 4$; *and*

(4) $k \in \{0,1,2,\cdots,v^2-7,v^2-6,v^2-4,v^2\}$ *when* $v \geq 5$.

From Theorem 2.1 we find that for order 2 and 3 there is no large set of latin squares with one entry in common. Therefore in what follows we will consider the latin squares of order n for $n \geq 4$.

It is easy to see that if two latin squares have one entry in common then we can use a row permutation and a column permutation such that this common entry is in the (1,1)-th cell. Without loss of generality, we let this entry be 1, and denote $D(n,1)$ by $D(n)$. Thus it is easy to get the following result.

Lemma 2.2. *For any positive integer n, $D(n) \leq n-1$.*

Proof. Since the (1,1)-th entry of all the latin squares we discuss should be 1, the only way we can fill out the (1,2)-th entry of the large set of latin squares with one common entry must be one of the integers in $\{2,3,\cdots,n\}$. Since they must be different, therefore $D(n) \leq n-1$. ∎

Next we will give a construction for those $D(n)$ latin squares of order n with one fixed entry in common, for each $n \geq 4$.

Theorem 2.3. *For any integer n, $n \geq 4$, there are $n-1$ latin squares of order n such that any pair of them have one entry in common, i.e. $D(n) = n-1$.*

Proof. (Since the proof will be by direct construction, it is helpful if the reader can look at the example following the proof step by step.) Let $M = [m_{i,j}]$ be an idempotent latin square of order $n-1$ based on the set $\{2,3,\cdots,n\}$. Let $P = [p_{i,j}]$ be defined as follows:

(1) $p_{1,j} = j$, for $j = 1,2,\cdots,n$.

(2) $p_{i,1} = i$, for $i = 2,3,\cdots,n$.

(3) $p_{i+1,j+1} = m_{i,j}$, if $i \neq j$, $i,j = 1,2,\cdots,n-1$, and

(4) $p_{i,i} = 1$, for $i = 1,2,\cdots,n$.

Then P is a latin square based on the set $\{1,2,\cdots,n\}$. Let F_t be a function from $\{2,3,\cdots,n\}$ to $\{1,2,\cdots,n\}$ defined as $F_t(j) = p_{t,j}$, for each $j = 2,3,\cdots,n$ and $t = 2,3,\cdots,n$. Then from P and F_t we can get $L(t) = [l(t)_{i,j}]$, for each $t \in \{2,3,\cdots,n\}$ defined as follows:

(1) $l(t)_{1,1} = l(t)_{t,t} = 1$.

(2) $l(t)_{i,j} = F_t(p_{i,j})$, for $i \neq j$, $i,j = 2,3,\cdots,n$.

(3) $l(t)_{1,j} = F_t(j)$, for $j = 2,3,\cdots,n$, and $j \neq t$.

(4) $l(t)_{i,1} = l(t)_{1,i}$, for $i = 2,3,\cdots,n$.

and

(5) $l(t)_{i,i} = p_{1,t}$, for $i \neq t$, $i = 2,3,\cdots,n$.

From the above construction, it is easy to check that P is a latin square of order n based on the set $\{1, 2, \cdots, n\}$. By the properties of a latin square we know that F_t is a one-to-one function from $\{2, 3, \cdots, n\}$ to $\{1, 2, \cdots, n\}$ for each t, and F_2, F_3, ..., F_n are one-to-one functions such that $F_i(x) \neq F_j(x)$ for each $i \neq j$ and $x \in \{2, 3, \cdots, n\}$.

Next we will show that $L(t)$ is a latin square of order n, for each $t \geq 2$.

(1) For $i \neq j$ and $i \neq k$, where $i, j, k \in \{2, 3, \cdots, n\}$, if $l(t)_{i,j} = l(t)_{i,k}$ then $F_t(p_{i,j}) = F_t(p_{i,k})$, this implies that $p_{i,j} = p_{i,k}$. Thus $j = k$, since P is a latin square.

(2) If $l(t)_{i,j} = l(t)_{i,i}$ then $F_t(p_{i,j}) = p_{1,t}$. From the definition of F_t, we know that $F_t(p_{i,j}) = p_{t,k}$ where $k = p_{i,j}$. Since $p_{1,t} = t = p_{t,1}$, we get $p_{t,k} = p_{t,1}$, implies $k = 1$, i.e. $p_{i,j} = 1 = p_{i,i}$. Therefore $i = j$.

(3) If $l(t)_{1,j} = l(t)_{1,k}$ and $j, k > 1$, then $F_t(j) = F_t(k)$, and hence $j = k$.

From (1), (2) and (3), we know that there is no symbol occurring twice in any row of $L(t)$. Similarly, we can prove that there is no symbol occurring twice in any column. Thus $L(t)$ is a latin square of order n. Since F_2, F_3, ..., F_n are $n-1$ distinct functions, $L(2)$, $L(3)$, $\cdots$, $L(n)$ are disjoint latin squares except the (1,1)-th entry. Therefore there are $n-1$ latin squares of order n such that any pair of them have one entry in common, for each $n \geq 4$. ∎

Example. Using the construction above, we can construct a large set of latin squares of order 4 with one entry in common. Let M be defined as follows:

$$M = \begin{array}{|ccc|} \hline 2 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 4 \\ \hline \end{array}$$

Then we can construct P:

$$P = \begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ \hline \end{array}$$

By using P, we can generate three functions from $\{2, 3, 4\}$ to $\{1, 2, 3, 4\}$ and obtain three latin squares, respectively.

$$F_2(2) = 1,\ F_2(3) = 4,\ F_2(4) = 3$$

$$L(2) = \begin{array}{|cccc|} \hline 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ \hline \end{array}$$

$$F_3(2) = 4,\ F_3(3) = 1,\ F_3(4) = 2$$

$$L(3) = \begin{array}{|cccc|} \hline 1 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 2 & 1 & 4 & 3 \\ \hline \end{array}$$

$$F_4(2) = 3,\ F_4(3) = 2,\ F_4(4) = 1$$

$$L(4) = \begin{array}{|cccc|} \hline 1 & 3 & 2 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 2 & 3 & 1 \\ \hline \end{array}$$

Then it is easy to check that $L(2)$, $L(3)$ and $L(4)$ have only one entry in common, mutually.

It seems that we can study this problem in a more general form by considering $k(\leq n)$ fixed common entries in the main diagonal of the latin squares of order $n \geq k$ such that $l_{i,i} = i$, $1 \leq i \leq k$. The solution will naturally generalize the result in idempotent latin squares which is obtained in [5].

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