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EXISTENCE OF SOLUTIONS OF IMPLICIT DIFFERENTIAL INCLUSIONS

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Abstract. We prove the existence of solutions for the implicit differential inclusions of the form

$$\dot{x}(t) \in A(t,x) + F(t,x,\dot{x}) \text{ for } t \in I, \quad \text{a.e.}$$
$$x(r) = 0$$

with $I = [r, r+a]$ by using the Covitz-Nadler fixed point theorem.

1. Introduction

The purpose of this paper is to prove the existence of solutions of implicit δ differential inclusions $\dot{x}(t) \in A(t,x) + F(t,x,\dot{x})$ where F is a setvalued function, Lipschitz continuous with respect to its last two variables and A is a single valued continuous function satisfying the Lipschitz condition. In [1] we assume the existence of the classical solutions of the above problem and proved that the solution set is a retract of $C(I,E)$ where E is a Banach space. In this paper we prove an existence theorem by using the Covitz-Nadler fixed point theorem.

Fixed point theorems are used as a tool to prove the existence of solutions of differential inclusions [2]. Papageorgiou [7] used the Schauder-Tichonoff fixed point theorem to prove the existence of solutions of the above problem with

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delay and without the term $A(t,x)$. Ricceri [8] and Kisielewicz [5] considered the above equation without the term $A(t,x)$ where as the former studied the classical solution sets with some applications while the latter proved the existence of solutions by using the Covitz-Nadler fixed point theorem.

2. Preliminaries

Let I be a compact interval of the real line and let $F : I \to 2^{R^n}$ be a setvalued function. Denote by $S(F)$ the collection of all L-integrable (Lebesgue integrable) functions $u : I \to R^n$ having the property that $u(t) \in F(t)$ a.e. in I. The set $S(F)$ is said to be the trajectory integrals of the setvalued function F. Especially interesting, from the point of view of functional differential inclusions, are subtrajectory integrals of such family of setvalued functions $F(z) : I \to 2^{R^n}$ depending on a parameter z. The subtrajectory integrals of such family setvalued functions will be denoted by $S(F)(z)$.

Let $L(I,R^n)$ be the space of all L-integrable functions $u : I \to R^n$. A setvalued map $F : I \to Comp(R^n)$ (non empty compact subsets of R^n) is said to be integrably bounded if there exists a non negative function $u \in L(I,R^n)$ such that $F(t) \subset u(t)B$ a.e. in I, where B is a closed unit ball in R^n. Let $(\mathcal{A}(I,R^n), d)$ be the metric space of all setvalued functions $F : I \to Comp(R^n)$ that are L-measurable (Lebesgue) and integrably bounded with a metric d defined by

$$d(F,G) = \int_I H(F(t),G(t)) \text{ for } F,G \in \mathcal{A}(I,R^n).$$

where the Hausdorff metric H is defined by

$$\begin{aligned} H(A,B) &= \max\{\overline{H}(A,B), \overline{H}(B,A)\} \text{ where} \\ \overline{H}(B,A) &= \max_{b\in B}\{dist(b,A), dist(b,A)\} = \inf_{a\in A} |b-a| \end{aligned}$$

We shall consider a family of all setvalued functions $F : I \times C(I,R^n) \times L(I,R^n) \to Comp(R^n)$ such that $F(\cdot,x,z) \in \mathcal{A}(I,R^n)$ for each fixed $(x,z) \in C(I,R^n) \times L(I,R^n)$.

We say that the setvalued map F is Lipschitz continuous with respect to its last two variables if there is an L-integrable function $k : I \to R$ such that for every $x, \overline{x} \in C(I, R^n)$, $z, \overline{z} \in L(I, R^n)$ and a.e. $t \in I$ one has

$$H(F(t,x,z), F(t,\overline{x},\overline{z}) \le k(t) \max(|x-\overline{x}|_t, |z-\overline{z}|_t)$$

where

$$\begin{aligned} |x-\overline{x}|_t &= \sup_{r \le s \le t} |x(s) - \overline{x}(s)| \text{ and} \\ |z-z|_t &= \int_r^t |z(s) - \overline{z}(s)| ds. \end{aligned}$$

We also use the norm $\| \ \|_0$ as $\|G(t)\|_0 = H(G(t), \{0\})$.

We prove our existence theorem by using the following fixed point theorem.

Theorem (Covitz-Nadler). *Let (X, ρ) be a complete metric space and $G : X \to cl(X)$, (Closer of X), a setvalued contraction mapping, that is, such that $H(G(x), G(y)) \le k\rho(x,y)$ for every $x, y \in X$ with any $k \in [0,1)$. Then there exists $x \in X$ such that $x \in G(x)$.*

The existence of subtrajectory integrals in this paper are due to the continuous selection theorems of Fryszkowski [4] and Michael [6].

3. Existence

In this section we prove the existence of solutions of implicit differential inclusions of δ the form

$$\begin{aligned} \dot{x}(t) &\in A(t,x) + F(t,x,\dot{x}) \text{ for } t \in I, \text{ a.e.} \\ x(r) &= 0 \end{aligned} \tag{1}$$

with $I = [r, r+a]$. We assume the following hypotheses.

(i) $F : I \times C(I, R^n) \times L(I, R^n) \to Comp(R^n)$ is such that $F(\cdot, x, z) \in \mathcal{A}(I, R^n)$ for fixed (x, z) and F is Lipschitz continuous on $C(I, R^n) \times L(I, R^n)$ with respect to its last two variables.

(ii) $A : I \times C(I, R^n) \to R^n$ is single valued continuous function such that $|A(t,x) - A(t,y)| \leq k(t)|x-y|_t$ where $k : I \to R$ is a Lebesgue integrable function.

Let $T : L(I, R^n) \to C(I, R^n)$ be defined by

$$(Tu)(t) = \int_r^t u(s)ds, \text{ for } u \in L(I, R^n), \ t \in I.$$

Theorem. *Under the assumptions* (i) *and* (ii) *the initial value problem* (1) *has at least one solution.*

Proof. Let $F \circ T$ be defined on $I \times L(I, R^n)$ by setting $(F \circ T)(t, z) = F(t, Tz, z)$ for $(t, z) \in I \times L(I, R^n)$ and let $S(F \circ T)(z)$ the subtrajectory integrals of $(F \circ T)(\cdot, z)$ for each fixed $z \in L(I, R^n)$.

Now define the setvalued mapping $G(t, z)$ on $I \times L(I, R^n)$ to the compact subsets of R^n as

$$G(t, z) = A(t, Tz) + (F \circ T)(t, z).$$

Let

$$S(G)(z) = \{w \in L(I, R^n) : w(t) \in A(t, Tz) + S(F \circ T)(z)\}$$

be the subtrajectory integrals of $G(\cdot, z)$ for each fixed $z \in L(I, R^n)$. Clearly $G \in \mathcal{A}(I, R^n)$ and $S(G)$ is a nonempty closed and bounded subset of

$$L_G = \{w \in L(I, R^n) : |w(t)| \leq \|G(t, z)\|_0 \text{ for a.e. } t \in I\}$$

Let u be an arbitrary element of $S(F \circ T)(z)$. By Fillipov's theorem [3] there is a $v \in S(F \circ T)(\bar{z})$ such that

$$\begin{aligned}|u(t) - v(t)| &< dist(u(t), \ F(t, T\bar{z}, \bar{z})) \\ &\leq H(F(t, Tz, z), \ F(t, T\bar{z}, \bar{z}) \\ &\leq k(t) \max(|Tz - T\bar{z}|_t, \ |z - \bar{z}|_t)\end{aligned}$$

for $t \in I$ a.e.

Let $m = 3/M$, $M \in (0,1)$ and let us define the norm on $L(I; R^n)$ as

$$\begin{aligned} \|\omega\| &= \int_r^{r+a} e^{-mk(t)}|\omega(t)|dt \quad \text{for } \omega \in L(I,R^n) \text{ with} \\ K(t) &= \int_r^t k(s)ds \end{aligned}$$

Now

$$\begin{aligned} \overline{H}(S(G)(z), S(G)(\overline{z})) &= \overline{H}(A(t,Tz) + S(F\circ T)(z),\ A(t,T\overline{z}) + S(F\circ T)(\overline{z})) \\ &\leq \|A(t,Tz) - A(t,T\overline{z})\| + \|u - v\| \end{aligned}$$

for each $u \in S(F\circ T)(z)$ and any $v \in S(F\circ T)(\overline{z})$.

Now,

$$\begin{aligned} \|u - v\| &= \int_r^{r+a} e^{-mK(t)}|u(t) - v(t)|dt \\ &\leq \int_r^{r+a} k(t)\ e^{-mK(t)}\ \max(|Tz - T\overline{z}|_t,\ |z - \overline{z}|_t)dt \\ &\leq \int_r^{r+a} k(t)\ e^{-mK(t)} \sup_{r\leq s\leq t} |Tz(s) - T\overline{z}(s)|dt \\ &\quad + \int_r^{r+a} k(t)\ e^{-mK(t)}(\int_r^t |z(s) - \overline{z}(s)|ds)dt \end{aligned}$$

But,

$$\begin{aligned} &\int_r^{r+a} k(t)\ e^{-mK(t)} \sup_{r\leq s\leq t} |Tz(s) - T\overline{z}(s)|dt \\ &\leq \int_r^{r+a} k(t)\ e^{-mK(t)}(\int_r^t |z(s) - \overline{z}(s)|ds)dt \\ &= -(1/m)\ e^{-mK(r+a)} \int_r^{r+a} |z(s) - \overline{z}(s)|ds \\ &\quad + 1/m \int_r^{r+a} e^{-mK(t)}|z(t) - \overline{z}(t)|dt \\ &\leq \frac{M}{3}\|z - \overline{z}\| \end{aligned}$$

and

$$\int_r^{r+a} k(t)\ e^{-mK(t)}(\int_r^t |z(s) - \overline{z}(s)|ds)dt \leq \frac{M}{3}\|z - \overline{z}\|$$

Similarly,

$$\|A(t,Tz)-A(t,T\overline{z})\| = \int_r^{r+a} e^{-mK(t)}|A(t,Tz)-A(t,T\overline{z})|dt$$

$$\leq \int_r^{r+a} k(t)\ e^{-mK(t)}|Tz-T\overline{z}|_t dt$$

$$\leq \int_r^{r+a} k(t)\ e^{-mK(t)} \sup_{r\leq s\leq t} |Tz(s)-T\overline{z}(s)|dt$$

$$\leq \frac{M}{3}\|z-\overline{z}\|.$$

Therefore,

$$\overline{H}(S(G)(z),\ S(G)(\overline{z})) \leq M\|z-\overline{z}\|$$

In a similar way we obtain

$$\overline{H}(S(G)(\overline{z}),\ S(G)(z)) \leq M\|z-\overline{z}\|.$$

Hence $H(S(G)(z),\ S(G)(\overline{z})) \leq M\|z-\overline{z}\|$ for each $z,\overline{z}\in L(I,R^n)$.

Put $X = L(I,R^n)$ and $\rho(z_1,z_2) = \|z_1-z_2\|$ for every $z_1,z_2\in X$. Obviously (X,ρ) is a complete metric space. Since $M\in(0,1)$ and $S(G)(z)\in cl(L(I,R^n))$ for every $z\in X$, $S(G)$ is setvalued contractive mapping of X into $cl(X)$. Now by Covitz-Nadler fixed point theorem there is $z\in X$ such that $z\in S(G)(z)$. Put $x = Tz$. Then we obtain $x(r) = 0$ and $\dot{x}(t)\in A(t,x)+F(t,x,\dot{x})$ for a.e. $t\in I$. Therefore x is a solution of (1).

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