TAMKANG JOURNAL OF MATHEMATICS
Volume 24, Number 3, Autumn 1993

ON THE SOLUTION OF EQUATIONS WITH NONDIFFERENTIABLE OPERATORS

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Abstract. We approximate solutions of equations with nondifferentiable operators using the Newton-Kantorovich method and the majorant theory. Under some as easy to verify assumptions as the ones given by Zabrejko and Nguen in [9] we improve their error estimates.

1. Introduction

Let X and Y be Banach spaces, and let $U(x_0, R)$ denote the closed ball with center $x_0 \in X$ and of radius R in X. Suppose that two operators F and G are defined on a convex subset D of X containing $U(x_0, R)$, with values in Y, where F is Fréchet differentiable at every interior point of $U(x_0, R)$ and satisfies the condition

$$\|F'(x+h) - F'(x)\| \leq A(r, \|h\|), x \in U(x_0, r), 0 \leq r \leq R, 0 \leq \|h\| \leq R - r, \quad (1)$$

while G satisfies the condition

$$\|G(x+h) - G(x)\| \leq B(r, \|h\|), x \in U(x_0, r), 0 \leq r \leq R, 0 \leq \|h\| \leq R - r. \quad (2)$$

Here A, B are nonnegative and continuous functions of two variables such that if one of the variables is fixed then A, B are non-decreasing functions of the other on the interval $[0, R]$. Moreover, the following are true:

Received April 2, 1990.
(1980) A. M. S. classification codes: 49D15, 47H17, 65.
Keywords and phrases: Banach space, Newton's method, Banach's, closed graph theorem.

(a) the function $\frac{\partial A(0,t)}{\partial t}$ is positive, continuous and non-decreasing on $[0, R-r]$ with $A(0,0) = 0$; (3)

(b) B is linear in the second variable and the function $\frac{\partial B(R,t)}{\partial t}$ is positive, continuous and non-decreasing on $[0, R-r]$.

the definition of B implies that

$$B(r,t) \leq B(R,t), \text{ for all } 0 \leq r \leq R \text{ and } 0 \leq t \leq R-r. \tag{4}$$

Further, assume that the operator $F'(x_0)$ is invertible. We are concerned with approximating a solution x^* of the equation

$$F(x) + G(x) = 0 \tag{5}$$

in $U(x_0, R)$ using the approximations

$$z_{n+1} = z_n - F'(x_0)^{-1}(F(z_n) + G(z_n)), \; z_0 = x_0, n = 0, 1, 2, \cdots \tag{6}$$

and

$$x_{n+1} = x_n - F'(x_n)^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, 2, \cdots. \tag{7}$$

Equation (5) has been studied extensively in the case when $G = 0$, using the modified iteration (6) or the Newton-Kantorovich iteration (7), [2], [3], [4],[5]. In particular, Potra and Ptâk have obtained elegant error estimates by means of a method based on a special variant of Banach's closed graph theorem [1], [4], [8]. Zincenko in [10] and Zabrejko and Ngyen in [9] studied the case $G \neq 0$ under the hypotheses (1) and (2) provided that A, B are given by

$$A(r,t) = k(r)t \tag{8}$$

and

$$B(r,t) = \epsilon(r)t, \tag{9}$$

where $k(r)$ and $\epsilon(r)$ are non-decreasing functions on the interval $[0, R]$. Further work on equation (5) can be found in [8].

In this paper we show that under very similar assumptions our error estimates on the distances $\|x^* - x_n\|$, $\|x_n - x_{n+1}\|$ ($\|x^* - z_n\|$, $\|z_n - z_{n+1}\|$) are better than the ones given in [9].

We will assume that for a fixed $r \in [0, R]$ the functions A and B can be extended such that $\|h\| \in [0, r]$.

2. Existence Theorems

We will need to define the constants

$$a = \|F'(x_0)^{-1}(F(x_0) + G(x_0))\|, \quad b = \|F'(x_0)^{-1}\| \tag{10}$$

and the functions

$$\omega(r) = \int_0^r k(t)dt, \tag{11}$$

$$\varphi(r) = a + b\int_0^r \omega(t)dt - r, \tag{12}$$

$$\varphi_\gamma(r) = a + b\int_0^r A(0,t)dt - r, \tag{13}$$

$$\psi(r) = b\int_0^r \epsilon(t)dt, \tag{14}$$

$$\psi_\gamma(t) = bB(R,t), \tag{15}$$

$$\chi(r) = \varphi(r) + \psi(r) \tag{16}$$

and

$$\chi_\gamma(r) = \varphi_\gamma(r) + \psi_\gamma(r). \tag{17}$$

As in [9], the main advantage of our approach consists in the fact that the study of equation (5) reduces to the study of a simple scalar equation

$$\chi_\gamma(r) = 0 \quad \text{on} \quad [0, R]. \tag{18}$$

The following theorem justifies our claim.

Theorem 1. *Suppose that the function $\chi_\gamma(r) = \varphi_\gamma(r) + \psi_\gamma(r)$ has a unique zero s in the interval $[0, R]$, and $\chi_\gamma(R) \leq 0$. Then equation (5) has a solution*

x^* in $U(x_0, r)$. This solution is unique in $U(x_0, s)$, and the iterates generated by (6) are well-defined for all n, belong to $U(x_0, s)$, and satisfy the estimates

$$\|z_{n+1} - z_n\| \leq \overline{s}_{n+1} - \overline{s}_n, \quad n = 0, 1, 2, \cdots \tag{19}$$

and

$$\|z_n - x^*\| \leq s - \overline{s}_n, \quad n = 0, 1, 2, \cdots. \tag{20}$$

Moreover, the sequence defined by

$$\overline{s}_{n+1} = d(\overline{s}_n), \quad n = 0, 1, 2, \cdots, \overline{s}_0 = 0 \tag{21}$$

where

$$d_\gamma(r) = r + \chi_\gamma(r) \tag{22}$$

is monotonically increasing and converges to s.

Proof. Let us define the sequence

$$z_{n+1} = P(z_n), \quad n = 0, 1, 2, \cdots, \ z_0 = x_0 \tag{23}$$

where

$$P(z) = I - F'(x_0)^{-1}(F(z) + G(z)). \tag{24}$$

Moreover, consider the numerical sequence given by (21).

As in [9, p.675], a simple geometrical argument shows that the sequence (21) is monotonically increasing and convergent to s.

We shall show the estimate

$$\|z_{n+1} - z_n\| \leq \overline{s}_{n+1} - \overline{s}_n. \tag{25}$$

For $n = 0$ the inequality (25) is true since

$$\|z_1 - z_0\| = \|F'(x_0)^{-1}(F(x_0) + G(x_0))\| = a = d(0) = \overline{s}_1 - \overline{s}_0.$$

Let us assume that (25) holds for $n = 0, 1, 2, \cdots, k-1$. Then

$$\begin{aligned} &\|z_{k+1} - z_k\| = \|P(z_k) - P(z_{k-1})\| \\ &\leq \|z_k - z_{k-1} - F'(x_0)^{-1}(F(z_k) - F(z_{k-1}))\| + \|F'(x_0)^{-1}(G(z_k) - G(z_{k-1}))\| \\ &\leq b \int_0^1 \|F'((1-t)z_{k-1} + tz_k) - F'(x_0)\| \, \|z_k - z_{k-1}\| dt + b\|G(z_k) - G(z_{k-1})\|. \end{aligned} \tag{26}$$

Set

$$h = (1-t)z_{k-1} + tz_k - (1-t)x_0 - tx_0,$$

then

$$\|h\| \le (1-t)\|z_{k-1} - x_0\| + t\|z_k - x_0\| \le (1-t)\bar{s}_{k-1} + t\bar{s}_k. \tag{27}$$

Using (1), (2), (4), (26), (27) and the induction hypothesis, (26) becomes

$$\|z_{k+1} - z_k\| \le b\int_{\bar{s}_{k-1}}^{\bar{s}_k} A(0,t)dt + bB(\bar{s}_{k-1}, \|z_k - z_{k-1}\|)$$

$$\le b[\int_{\bar{s}_{k-1}}^{\bar{s}_k} A(0,t)dt + bB(R, \bar{s}_k - \bar{s}_{k-1})] = d_\gamma(\bar{s}_k) - d_\gamma(\bar{s}_{k-1}) = \bar{s}_{k+1} - \bar{s}_k,$$

which shows (25) for $n = k$.

We have shown that (21) is a Cauchy sequence in a Banach space and as such it converges to a solution $x^* \in U(x_0, s)$. We will now show that x^* is a unique solution of equation (5) in $U(x_0, s)$. Let us define the sequences

$$y_{n+1} = P(y_n), \quad n = 0,1,2,\cdots \ y_0 \in U(x_0, R) \tag{28}$$

and

$$s_{n+1}^+ = d_\gamma(s_n^+), \quad n = 0,1,2,\cdots, \quad s_0^+ = R. \tag{29}$$

To show uniqueness it suffices to show the estimates

$$\|z_n - y_n\| \le s_n^+ - s_n^-, \quad n = 0,1,2,\cdots. \tag{30}$$

Using the same geometrical argument as in [9, p. 675] we can show that the sequence (29) is monotonically decreasing and convergent to s. That is, if for y_0 we choose the second solution $x_1^* \in U(x_0, r)$ of equation (5), we get by (30), that $\|x_1^* - x^*\| \le s_n^+ - s_n^-$. That is, $x_1^* = x^*$. For $n = 0$, (30) is true since $\|y_0 - z_0\| \le R - 0 = R$. Let us assume that (30) holds for $n = 0,1,2,\cdots,k$.

Using (1) and (2)

$$\|y_{k+1}-z_{k+1}\| = \|p(y_k)-P(z_k)\|$$

$$\leq \|y_k-z_k-F'(x_0)^{-1}(F(y_k)-F(z_k))\|+\|F'(x_0)^{-1}(G(y_k)-G(z_k))\|$$

$$\leq b\int_0^1 \|F'((1-t)z_k+ty_k)-F'(x_0)\|\cdot\|y_k-z_k\|dt+bB(R,s_k^+-s_k^-)$$

$$\leq b\int_0^1 A(0,(1-t)s_k^-+ts_k^+)(s_k^+-s_k^-)dt+bB(R,s_k^+-s_k^-)$$

$$\leq b\int_{s_k^-}^{s_k^+} A(0,t)dt+bB(R,s_k^+-s_k^-)=d_\gamma(s_k^+)-d_\gamma(s_k^-)=s_{k+1}^+-s_{k+1}^-.$$

That completes the proof of the theorem.

We can now prove the main theorem.

Theroem 2. *Assume*

(i) *the hypotheses of Theorem* 1 *are ture;*

(ii) *the number* R *defined in Theorem* 1 *is such that*

$$1-bA(0,R)>0$$

and

$$T(R)\leq R$$

where

$$T(r) = a+\frac{b\int_0^r A(r,t)dt+bB(r,r)}{1-bA(0,r)}, \quad r\in[0,\infty).$$

Then

(a) the sequence $\{\overline{\rho}_n\}$, $n=0,1,2,\cdots$ given by

$$\overline{\rho}_0 = 0, \quad \overline{\rho}_1 = a,$$

$$\overline{\rho}_{n+1} = \overline{\rho}_n-\frac{b\int_{\overline{\rho}_{n-1}}^{\overline{\rho}_n} A(\overline{\rho}_{n-1},t)dt+\psi_\gamma(\overline{\rho}_n)-\psi_\gamma(\overline{\rho}_{n-1})}{\varphi'_\gamma(\overline{\rho}_n)}, \quad n=1,2,\cdots \quad (31)$$

is bounded above by R, it is monotonically increasing and converges to some s^* with $s \leq s^* \leq R$. The sequence $\{q_n\}$ given by

$$q_{n+1} = q_n - \frac{\chi_\gamma(q_n)}{\varphi'_\gamma(q_n)}, \quad q_0 = 0, \quad n = 0, 1, 2, \cdots$$

is monotonically increasing and converges to s. Moreover

$$q_{n+1} - q_n \leq \overline{\rho}_{n+1} - \overline{\rho}_n,$$

$$s - q_n \leq s^* - \overline{\rho}_n$$

and

$$q_n \leq \overline{\rho}_n \quad \text{for all} \quad n = 0, 1, 2, \cdots.$$

(b) The iterates generated by (7) are well-defined in $U(x_0, s^*)$ for all n, and satisfy the estimates

$$\|x_{n+1} - x_n\| \leq \overline{\rho}_{n+1} - \overline{\rho}_n, \quad n = 0, 1, 2, \cdots \tag{32}$$

and

$$\|x_{n+1} - x^*\| \leq s^* - \overline{\rho}_n, \quad n = 0, 1, 2, \cdots. \tag{33}$$

Proof. (a) We will first show that the sequence given by (31) is bounded above by R. This is true for $k = 0, 1$. For $k = 2$ we have

$$\overline{\rho}_2 = a + \frac{b \int_0^a A(0, t)dt + bB(R, a)}{1 - bA(0, a)} \leq T(R) \leq R.$$

Let us assume that $\overline{\rho}_k \leq R$ for $k = 0, 1, 2, \cdots, n$. We will show that $\overline{\rho}_k \leq R$ for $k = n + 1$. The sequence given by (31) is such that

$$\begin{aligned}
\overline{\rho}_{n+1} &\leq \overline{\rho}_n + \frac{b \displaystyle\int_{\overline{\rho}_{n-1}}^{\overline{\rho}_n} A(R, t)dt + bB(R, \overline{\rho}_n - \overline{\rho}_{n-1})}{1 - bA(0, R)} \\
&\leq \cdots \leq a + \frac{b[\displaystyle\int_{\overline{\rho}_{n-1}}^{\overline{\rho}_n} A(R, t)dt + \int_{\overline{\rho}_1}^{\overline{\rho}_2} A(R, t)dt + \cdots + \int_{\overline{\rho}_{n-1}}^{\overline{\rho}_n} A(R, t)dt]}{1 - bA(0, R)} \\
&\quad + \frac{b(B(R, \overline{\rho}_1 - \overline{\rho}_0 + \overline{\rho}_2 - \overline{\rho}_1 + \cdots + \overline{\rho}_n - \overline{\rho}_{n-1})}{1 - bA(0, R)} \\
&= T(R) \leq R, \quad \text{for all} \quad n = 0, 1, 2, \cdots,
\end{aligned}$$

by hypothesis. Using (31) it can easily be seen that the sequence $\{\overline{\rho}_n\}$ is monotonically increasing. Hence, it converges to some s^* with $a \le s^* \le R$. The same proof as the proof of Proposition 3 in [9, p.677] shows that the sequence $\{q_n\}$ is monotonically increasing and converges to s. It can easily be seen using induction on n that $\varphi'(q_n) \le \varphi'(\overline{\rho}_n)$ for all $n = 0, 1, 2, \cdots$. Using the definitions of the sequences $\{q_n\}$ and $\{\overline{\rho}_n\}$ and the above inequality we can now easily deduce the rest of the results in (a).

(b) As in [9, p.678] for $n = 0$, (32) is true. Suppose that (32) is ture for $n = 0, 1, 2, \cdots, k-1$. We must show that $F'(x_k)$ is invertible. We have

$$\|x_k - x_0\| \le \sum_{j=1}^{k} \|x_j - x_{j-1}\| \le \sum_{j=1}^{k} (\overline{\rho}_j - \overline{\rho}_{j-1}) = \overline{\rho}_k \tag{34}$$

(by the induction hypothesis).

By (1) we obtain

$$\|F'(x_0)^{-1}(F'(x_k) - F'(x_0))\| \le bA(0, \overline{\rho}_k) < bA(0, s^*) = \varphi'_\gamma(s^*) + 1 \le 1.$$

By the Banach lemma on invertible operators $F'(x_k)$ is also invertible and

$$\|F'(x_k)^{-1}\| \le \frac{b}{1 - bA(0, \overline{\rho}_k)} = -\frac{b}{\varphi'_\gamma(\overline{\rho}_k)}. \tag{35}$$

Using (7) we get

$$\begin{aligned} \|x_{k+1} - x_k\| &= \|F'(x_k)^{-1}(F(x_k) + G(x_k))\| \\ &= \|F'(x_k)^{-1}(F(x_k) - F(x_{k-1}) - F'(x_{k-1}(x_k - x_{k-1}) + G(x_k) - G(x_{k-1}))\| \\ &\le \|F'(x_k)^{-1}\| [\int_0^1 \|F'((1-t)x_{k-1} + tx_k) - F'(x_{k-1})\| \cdot \|x_k - x_{k-1}\| dt \\ &\quad + \|G(x_k) - G(x_{k-1})\|]. \end{aligned} \tag{36}$$

By (1), (2), (34) and (35), (36) becomes,

$$\begin{aligned} \|x_{k+1} - x_k\| &\le -\frac{b}{\varphi'_\gamma(\overline{\rho}_k)} \int_0^1 A(\overline{\rho}_{k-1}, (1-t)\overline{\rho}_{k-1} + t\overline{\rho}_k)(\overline{\rho}_k - \overline{\rho}_{k-1}) dt \\ &\quad - \frac{1}{\varphi'_\gamma(\overline{\rho}_k)} (\varphi_\gamma(\overline{\rho}_k) - \psi_\gamma(\overline{\rho}_{k-1})) \\ &= -\frac{b}{\varphi'_\gamma(\overline{\rho}_k)} \int_{\overline{\rho}_{k-1}}^{\overline{\rho}_k} A(\overline{\rho}_{k-1}, t) dt - \frac{1}{\varphi'_\gamma(\overline{\rho}_k)} (\psi_\gamma(\overline{\rho}_k) - \psi_\gamma(\overline{\rho}_{k-1})) \\ &= \overline{\rho}_{k+1} - \overline{\rho}_k. \end{aligned}$$

That shows (32) for $n = k$. The estimates (33) follow now immediately using (32).

That completes the proof of the theorem.

In Theorem 2 we really wanted to show that the sequence $\{q_n\}$ majorizes the sequence $\{x_n\}$. Instead we showed the results for the sequence $\{\overline{\rho}_n\}$.

The corresponding estimates to (19), (20), (32) and (33) in [9] are given by

$$\|z_{n+1} - z_n\| \leq \rho^*_{n+1} - \rho^*_n, \tag{37}$$

$$\|z_n - z^*\| \leq \rho - \rho^*_n, \tag{38}$$

$$\|x_{n+1} - x_n\| \leq \rho_{n+1} - \rho_n, \tag{39}$$

and

$$\|x_n - x^*\| \leq \rho - \rho_n \tag{40}$$

where ρ is the unique zero of $\chi(r) = \varphi(r) + \psi(r)$ in $[0, R]$ and the sequences $\{\rho^*_n\}$ and $\{\rho_n\}$ are given by

$$\rho^*_{n+1} = d(\rho^*_n), \quad n = 0, 1, 2, \cdots, \ \rho^*_0 = 0, \tag{41}$$

where

$$d(r) = r + \chi(r) \tag{42}$$

and

$$\rho_{n+1} = \rho_n - \frac{\chi(\rho_n)}{\varphi'(\rho_n)}, \quad n = 0, 1, 2, \cdots, \ \rho_0 = 0. \tag{43}$$

We will now show that under simple assumptions

$$\overline{\rho}_{n+1} - \overline{\rho}_n \leq \rho_{n+1} - \rho_n.$$

We can show similiarly that the rest of our error estimates (19), (20) and (33) are better than (37), (38) and (40) respectively.

We will first need to state a theorem whose proof can be found in [9, p.673].

Theorem 3. *Suppose that the function $\chi(r) = \varphi(r) + \psi(r)$ has a unique zero ρ in the interval $[0, R]$, and $\chi(R) \leq 0$. Then equation (5) admits a solution x^* in*

Proof. (a) We will only show (48). The rest will follow similarly. By (12)-(17), (44) and (45) we obtain immediately that

$$\chi_\gamma(r) \leq \chi(r) \quad \text{for all} \quad r \in [0, R]. \tag{55}$$

Moreover, using (44) we get

$$-\varphi'(r) = 1 - b\omega(r) \geq 1 - bA(0, r) > 0. \tag{56}$$

That is

$$-\frac{1}{\varphi'(r)} \leq -\frac{1}{\varphi'_\gamma(r)}. \tag{57}$$

The results now follows from (55), (57), (31) and (43).

(b) By Theorems 2 and 3 to show (53) we only need to show

$$\overline{\rho}_{n+1} - \overline{\rho}_n \leq \rho_{n+1} - \rho_n \quad \text{for all} \quad n = 0, 1, 2, \cdots. \tag{58}$$

By (41) and (43), inequality (58) will be true if

$$b\int_{\overline{\rho}_{n-1}}^{\overline{\rho}_n} A(\overline{\rho}_{n-1}, t)dt + bB(R, \overline{\rho}_n - \overline{\rho}_{n-1}) \leq b\int_0^{\rho_n} \omega(t)dt + a - \rho_n + b\int_0^{\rho_n} \epsilon(t)dt,$$

$$\text{for all} \quad n = 1, 2, \cdots \tag{59}$$

and

$$A(0, \overline{\rho}_n) \leq \omega(\rho_n), \quad \text{for all} \quad n = 0, 1, 2, \cdots. \tag{60}$$

The left hand side of (59) is bounded aobve by the left hand side of (51) for $n = 1$ and by the left hand side of (52) for $n \geq 2$. Whereas the right hand side of (50) is bounded below by the right hand side of (51) for $n = 1$ and the right hand side of (52) for $n \geq 2$. Hence (50) is true for all $n = 1, 2, \cdots$. Inequality (60) is certainly ture if inequality (50) is ture. That shows (53). Inequality (54) now follows easily. That completes the proof of the proposition.

Note that by Theorem 2, s^* can be replace with R in (50)-(52).

A discussion on the convergence speed of the numerical sequences (21) and (31) can now easily follow as the discussion in [9, p. 679-684]. However, we leave that to the motivated reader.

the ball $U(x_0, \rho)$, this solution is unique in $U(x_0, R)$, and the iterates generated by (6) and (7) are defined for all n, belong to $U(x_0, \rho)$ and satisfy the estimates (37)–(40). Moreover, the sequences $\{\rho_n^*\}$ and $\{\rho_n\}$ are monotonically increasing and convergent to ρ.

We can now justify the claim made at the introduction.

Proposition. *Under the hypotheses of Theorems 2 and 3,*

(a) *If*

$$A(0,t) \leq \omega(t), \quad 0 \leq t \leq R - r, \tag{44}$$

and

$$\psi_\gamma(r) \leq \psi(r), \quad 0 \leq r \leq R. \tag{45}$$

Then

$$\|z_{n+1} - z_n\| \leq \overline{s}_{n+1} - \overline{s}_n \leq \rho_{n+1}^* - \rho_n^*, \tag{46}$$

$$\|z_n - z^*\| \leq s - \overline{s}_n \leq \rho - \rho_n^*, \tag{47}$$

$$\overline{q}_{n+1} - q_n \leq \rho_{n+1} - \rho_n \tag{48}$$

and

$$s - q_n \leq \rho - \rho_n \quad \textit{for all} \quad n = 0, 1, 2, \cdots. \tag{49}$$

(b) *If*

$$A(0, s^*) \leq \omega(a), \tag{50}$$

$$b\int_0^a A(0,t)dt + bB(R,a) \leq b\int_0^a \omega(t)dt + b\int_0^a \epsilon(t)dt \tag{51}$$

and

$$b\int_a^{s^*} A(s^*,t)dt + bB(R, s^* - a) \leq b\int_0^{\rho_2} \omega(t)dt + a - \rho + b\int_0^{\rho_2} \in (t)dt. \tag{52}$$

Then

$$\|x_{n+1} - x_n\| \leq \overline{\rho}_{n+1} - \overline{\rho}_n \leq \rho_{n+1} - \rho_n \tag{53}$$

and

$$\|x_n - x^*\| \leq s^* - \overline{\rho}_n \leq \rho - \rho_n \quad \textit{for all} \quad n = 0, 1, 2, \cdots. \tag{54}$$

Remark. Let us define the function $\overline{\chi}_\gamma(r)$ by

$$\overline{\chi}_\gamma(r) = b\int_0^r A(r,t)dt + bB(R,r) + (a-r)(1-bA(0,r))$$

and the sequence $\{v_n\}$, by

$$v_{n+1} = v_n - \frac{\overline{\chi}_\gamma(v_n)}{\varphi'_\gamma(v_n)}, \quad v_0 = 0, \quad n = 1,2,\cdots. \tag{61}$$

Then, under the hypotheses of Theorem 2, $\overline{\chi}_\gamma(0) = a > 0$ and $\overline{\chi}_\gamma(R) \le 0$. That is, there exists a solution $s_1^* \in (0,R]$ of the equation $\overline{\chi}_\gamma(r) = 0$. Using induction on n, it is simple calculus to show that the seqence $\{v_n\}$, is monotonically increasing and converges to s_1^*. Moreover,

$$\overline{\rho}_n \le v_n$$

$$\|x_{n+1} - x_n\| \le v_{n+1} - v_n$$

$$\|x_n - x^*\| \le s^* - \rho_n \le s_1^* - v_n, \quad \text{for all} \quad n = 0,1,2,\cdots$$

and

$$s^* \le s_1^*.$$

Furthermore, if the following conditions are satisfied

$$\rho - a \le (v_2 - a)(1 - bA(0,R)), \tag{62}$$

$$A(r,t) \le \omega(t) \tag{63}$$

and

$$\psi_\gamma(t) \le \psi(t) \quad \text{for all} \quad r,t \in [0,R], \tag{64}$$

then

$$v_{n+1} - v_n \le \rho_{n+1} - \rho_n,$$

$$v_n \le \rho_n$$

and

$$s_1^* - v_n \le \rho - \rho_n \quad \text{for all} \quad n = 0,1,2,\cdots.$$

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