TAMKANG JOURNAL OF MATHEMATICS
Volume 24, Number 3, Autumn 1993

ON AN APPLICATION OF A VARIANT OF THE CLOSED GRAPH THEOREM AND THE SECANT METHOD

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Abstract. The method of nondiscrete mathematical induction is used to find error bounds for the Secant method. We assume only that the operator has Hölder continuous derivatives. In the case the Fréchet-derivative of the operator satisfies a Lipschitz condition our results reduce to the ones obtained by F. Potra (Num. Math. 1982).

Introduction

Consider the equation

$$f(x) = 0 \tag{1}$$

where f is a nonlinear operator mapping a subset E_f of a Banach space E_1 into another Banach space E_2.

Here we are concerned with finding solutions of (1) using the secant iterations

$$x_{n+1} = x_n - \delta f(x_{n-1}, x_n)^{-1} f(x_n) \tag{2}$$

$$x_{n+1} = x_n - \delta f(x_{-1}, x_0)^{-1} f(x_n) \tag{3}$$

where x_{-1} and x_0 are two points in the domain of f, and δf is a consistent approximation of f'.

Received April 11, 1990; revised March 16, 1993.
(1980) A.M.S. classification codes: 65J15, 65B05, 65L50, 65M50, 46B15.
Key words and phrases: Nondiscrete mathematical induction, Secant method, Banach space, Hölder continuous derivatives.

This work is based upon the elegant work of F. Potra included in [5] concerning the error analysis of the Secant method. One of Potra's basic assumptions is the fact that essentially the linear operator f' is Lipschitz continuous. However, in the presence of some interesting examples (see part (III)), where f' is only Hölder continuous, we extend most of the results contained in [5] for the iteration (2). We leave the extension of the results for (3) to the motivated reader.

We furnish two examples in part (III) to show that our results can be applied where as the equivalent results in [5] cannot.

Since our results are drawn almost in the same lines with the ones in [5], we will need to restate some here.

Note that the results of this paper could be obtained by using one dimensional rate of convergence only, as was done in Potra [5], and not the more general rates of convergence introduced in Potra [6].

1. Preliminaries

Consider a class C of pairs (f, v_0) where f is as above and $v_0 = (x_{-k+1}, \cdots, x_0)$ is a system of k points from E_f. We want to attach to each pair $(f, v_0) \in C$ a sequence $\{x_n\}$, $n = 0, 1, 2, \cdots$ of points of E_f converging to a root x^* of (1). To achieve this we associate with the pair (f, v_0) an operator $F : E \subset E_f^p \to E_1$, where $k \geq p$ and try to obtain a sequence $\{x_n\}$, $n = 0, 1, 2, \cdots$, by the scheme:

$$x_{x+1} = F(x_{n-p+1}, \cdots, x_n), \quad n = 0, 1, 2, \cdots. \tag{4}$$

The above scheme will yield a sequence $\{x_n\}$, $n = 0, 1, 2, \cdots$, if $u_0 = (x_{-p+1}, \cdots, x_0)$ is an admissible system of starting points in the sense given by the following defintion:

Definition 1. Consider an operator $F : E \subset E_1^p \to E_1$ and define recursively

$$\tilde{E}_0 = E, \ \tilde{E}_{n+1} = \{u = (y_1, \cdots, y_p) \in \tilde{E}_n; (y_2, \cdots, y_p, F(u)) \in \tilde{E}_n\}, \quad n = 0, 1, 2.$$

Any $u_0 \in E_\infty = \bigcap_{n \geq 0} \tilde{E}$ will be called an admissible system of starting points for the scheme (4).

If u_0 is an admissible system of starting points for the scheme (4), we shall say that (4) is well defined.

Definition 2. Let C be a class of pairs (f, v_0) where f is a nonlinear operator defined on a subset E_f of a Banach space E_1 with values in a Banach space E_2, and $v_0 = (x_{-k+1}, \cdots, x_0) \in E_f^k$. Let $p \leq k$. By an iterative procedure of type $(p; 1)$ for the class C, we mean an application which associates with any $(f, v_0) \in C$ an operator $F : E \subset E_f^p \subset E_1$ having the following two properties:

(i) $u_0 = (x_{-p+1}, \cdots, x_0)$ is an admissible system of starting points for the scheme (4);

(ii) the sequence $\{x_n\}$, $n = 0, 1, 2, \cdots$ given by (4) converges to a root x^* of (1).

Having an iterative procedure of type $(p; 1)$ for the class C it is important to find a function $\alpha : \mathbf{Z}_+ \to \mathbf{R}_+$ and a function $\beta : \mathbf{R}_+^p \to \mathbf{R}_+$ such that the following inequalities are satisfied

$$d(x_n, x^*) \leq \alpha(n) \tag{5}$$

$$d(x_n - x^*) \leq \beta(d(x_{n-p+1}, x_{n-p}), \cdots, d(x_n, x_{n-1})) \tag{6}$$

for every pair $(f, x_0) \in C$ and every positive integer n.

The inequalities (5) are called apriori estimates because the right hand side can be computed before obtaining the points $x_1, \cdots, x_n$ via (4), while the inequalities (3) are called aposteriori estimates because their right hand side can be computed only after obtaining these points.

The estimates (5) and/or (6) will be called sharp if there exists a pair $(f, u_0) \in C$ for which these estimates are attained for all $n = 1, 2, 3, \cdots$.

In the study of (4), we use the nondiscrete mathematical induction. The method was initiated by V. Pták by refining the closed graph theorem [7], [8]. V. Pták used this method to investigate iterative algorithms of type (4) with $p = 1$. In [7] the method was extended for any p. See also [8] for the history of nondiscrete mathematical induction and its application to different gxeneralizations of the secant method.

Here we restate the results obtained in [6]. Let T denote either the set of all positive numbers, or an interval of the form

$$(0,b] = \{x \in \mathbb{R};\quad 0 < x \leq b\}.$$

Let ω be a mapping of the cartesian product T^p into T and let us consider the "iterates" $\omega^{(n)}$ of ω given for each $t = (t_1, \dots, t_p) \in T^p$ by the following scheme:

$$\omega^{(0)}(t) = t_p,\ \omega^{(n+1)}(t) = \omega^{(n)}(t_2, \cdots, t_p, \omega(t)), \quad n = 0,1,2,\cdots. \tag{7}$$

Definition 3. A mapping $\omega : T^p \to T$, with the above iteration law, is called a rate of convergence of type $(p;1)$ on T, if the series

$$\sigma(t) = \sum_{n=0}^{\infty} \omega^{(n)}(t) \tag{8}$$

is convergent for all $t \in T^p$.

From now on F will be a mapping of E into E_3, where E_3 is a complete metric space, and E a subset of the cartesian product E_3^p. We attach to F the mapping $\overline{F} : E \to E_3^p$, defined for every $u = (y_1, \dots, y_p) \in E$ by

$$\overline{F}(u) = (y_2, \dots, y_p, F(u)). \tag{9}$$

Denoting $u_n = (x_{n-p+1}, \dots, x_n)$ we have

$$u_{n+1} = \overline{F}(u_n), \quad n = 0,1,2,\cdots. \tag{10}$$

Similarly, we attach to ω the mapping $\overline{\omega} : T^p \to T^p$ defined by

$$\overline{\omega}(t) = (t_2, \dots, t_p, \omega(t)), \quad t = (t_1, \dots, t_p) \in T^p. \tag{11}$$

Denote by $\overline{\omega}^{(n)}$ the iterates of $\overline{\omega}$ in the sense of the usual composition of functions, that is

$$\overline{\omega}^{(0)}(t) = t, \quad \overline{\omega}^{(n+1)}(t) = \overline{\omega}(\overline{\omega}^{(n)}(t)).$$

Then (7) becomes

$$\omega^{(0)}(t) = t_p, \quad \omega^{(n+1)}(t) = \omega(\overline{\omega}^{(n)}(t)). \tag{12}$$

Finally, we introduce the notation

$$\beta(t) = \sigma(t) - t_p.$$

From (8) and (11) it follows that

$$\beta(t) = \sigma(\overline{\omega}(t)).$$

With the above notation we can state the following proposition whose proof can be found in [6] or [7].

Proposition 1. *Let E_3 be a complete metric space and let E be a subset of E_3^p. Let us consider the operators $F : E \to E_3$ and $Z : T^p \to \exp(E)$, where $\exp(E)$ denotes the class of all subsets of E. Let ω be a rate of convergence of type $(p;1)$ on T.*

If there exists $u_0 = (x_{-p+1}, \dots, x_0) \in E$ and $t_0 \in T^p$ such that

$$u_0 \in Z(t_0) \tag{13}$$

and if the relations

$$F(u) \in Z(\overline{\omega}(t)), \tag{14}$$

$$d(F(u), y_p) \leq t_p \tag{15}$$

are satisfied for all $t = (t_1, \dots, t_p) \in T^p$ and $u = (y_1, \dots, y_p) \in Z(t)$, then:

(i) *the iteration (4) is well defined.*

(ii) *There exists an $x^* \in E_3$ such that $x^* = \lim_{n\to\infty} x_n$.*

(iii) *The following relations are satisfied for all $n = 0, 1, 2, \cdots$*

$$u_n \in Z(\overline{\omega}^{(n)}(t_0)), \tag{16}$$

$$d(x_{n+1}, x_n) \leq \omega^{(n)}(t_0), \tag{17}$$

$$d(x_n, x_0) \leq \sigma(t_0) - \sigma(\overline{\omega}^{(n)}(t_0)), \tag{18}$$

$$d(x_n, x^*) \leq \sigma(\overline{\omega}^{(n)}(t_0)). \tag{19}$$

(iv) *Let n be a positive integer and let $d_n \in T^p$; if $u_{n-1} \in Z(d_n)$, then*

$$d(x_n, x^*) \leq \beta(d_n). \tag{20}$$

We will need the definition:

Definition 4. Let E_1 and E_2 be two Banach spaces and let E_4 be a subset of E_1. Let $f : E_4 \to E_2$ be a nonlinear operator which is Fréchet differentiable on E_4. We say that the Fréchet-derivative $f'(x)$ is Hölder continuous over E_4 if for some $c > 0$ and $q \in [0,1]$, and all $x, y \in E_4$

$$\|f'(x) - f'(y)\| \leq c\|x - y\|^q. \tag{21}$$

In this case, we say $f'(\cdot) \in H_{E_4}(c, q)$.

Definition 5. Let E_1 and E_2 be two Banach spaces and let E_4 be a convex subset of E_1. Let $f : E_4 \to E_2$ be a nonlinear operator which is Fréchet-differentiable on E_4. A mapping $\delta f : E_4 \times E_4 \to L(E_1, E_2)$, (the space of bounded linear operators from E_1 to E_2) will be called a consistent generalized approximation of f', if there exists a constant $d > 0$ such that

$$\|\delta f(x,y) - f'(z)\| \leq d(\|x - z\|^q + \|y - z\|^q), \quad q \in [0,1], \tag{22}$$

and for all $x, y, z \in E_4$.

The above condition implies the Hölder continuity of f'. Since,

$$\begin{aligned} \|f'(x) - f'(y)\| &= \|(f'(x) - \delta f(x,y)) + (\delta f(x,y) - f'(y))\| \\ &\leq d(\|x - x\|^q + \|y - x\|^q) + d(\|x - y\|^q + \|y - y\|^q) \\ &\leq 2d\|x - y\|^q. \end{aligned}$$

That is

$$\|f'(x) - f'(y)\| \leq c\|x - y\|^q, \quad c = 2d \text{ and for all } x, y \in E_4. \tag{23}$$

Also, as in [1], we can easily show that

$$\|f(x) - f(y) - f'(x)(x - y)\| \leq \frac{c}{1+q}\|x - y\|^{1+q} \tag{24}$$

for all $x, y \in E_4$.

Finally, for all $x, y, u, v \in E_4$ we have

$$\begin{aligned}
&\|f(u) - f(v) - \delta f(x,y)(u,v)\| \\
&= \|(f(u) - f(v) - f'(v)(u-v)) + (f'(v) - \delta f(x,y))(u-v)\| \\
&\leq \frac{2d}{1+q}\|u-v\|^{1+q} + d(\|x-v\|^q + \|y-v\|^q)\|u-v\| \qquad (25) \\
&\leq d\left(\frac{2}{1+q}\|u-v\|^q + \|x-v\|^q + \|y-v\|^q\right)\|u-v\|.
\end{aligned}$$

Let $C(h_0, q_0; r_0)$ be the class of all triplets (f, x_0, x_{-1}) satisfying the following properties:

(P_1) f is a nonlinear operator having the domain of definition E_f included into a Banach space E_1 and taking values in a Banach space E_2.

(P_2) x_0 and x_{-1} are two points of E_f such that

$$0 < \|x_0 - x_{-1}\| \leq q_0, \ \|x_0 - x_{-1}\| < \mu.$$

(P_3) f is Fréchet-differentiable in the open ball

$$U = U(x_0, \mu) = \{x \in E_f / \|x - x_0\| \leq \mu\}$$

and continuous on its closure $\overline{U}$.

(P_4) there exists a consistent generalized approximation δf of f' such that $D_0 := \delta f(x_{-1}, x_0)$ is invertible and

$$\|D_0^{-1}(\delta f(x,y) - f'(z)\| \leq h_0(\|x-z\|^q + \|y-z\|^q) \qquad (26)$$

for all $x, y, z \in U$ and some $h_0 \geq d \cdot \|D_0^{-1}\|$.

(P_5) the following inequality is satisfied:

$$\|D_0^{-1} f(x_0)\| \leq \overline{r}_0. \qquad (27)$$

(P_6) Assume that for each fixed $q \in [0,1]$, there exists a minimum positive number μ_0 such that the following conditions hold:

$$\alpha_0 = \frac{h_0 r_0^q (3+q)}{(1+q)[1 - h_0(2\mu_0^q + q_0^q)]} < 1, \quad r_0 = \max\{\overline{r}_0, q_0\} \qquad (28)$$

$$u_0 = \frac{r}{1-\alpha_0} \tag{29}$$

and

$$h_0\left(\frac{4}{q+1}\mu_0^q + q_0^q\right) < 1. \tag{30}$$

Define also the functions ω and σ on $T = [0, \mu_0]$ by

$$\omega(r) = \alpha_0 r \tag{31}$$

and

$$\sigma(r) = \frac{r}{1-\alpha_0}. \tag{32}$$

We will use the estimate

$$\begin{aligned}
&\|D_0^{-1}(f(u) - f(v) - \delta f(x,y)(u-v))\| \\
&= \|D_0^{-1}(f(u) - f(v) - f'(v)(u-v) + D_0^{-1}(f'(v) - \delta f(x,y))(u-v)\| \\
&\leq \frac{2d\|D_0^{-1}\|}{1+q}\|u-v\|^{1+q} + h_0(\|x-v\|^q + \|y-v\|^q)\|u-v\|
\end{aligned}$$

(by (24) and (26))

$$\leq h_0\left[\frac{2}{1+q}\|u-v\|^q + \|x-v\|^q + \|y-v\|^q\right]\|u-v\|. \tag{33}$$

2. Main results

Using (2) we shall show that if $(f, x_0, x_{-1}) \in C(h_0, q_0, r_0)$ then (1) has a solution x^* which is unique in a certain neighborhood of x_0.

We will need the following lemma whose proof can be obtained immediately by using relation (8).

Lemma 1. *The function ω given by (31) is a rate of convergence of type (1,1) on the interval $T = (0, \mu_0)$ and the corresponding σ-function is given by (32).*

We will now prove the main result.

Theorem 1. *If $(f, x_0, x_{-1}) \in C(h_0, q_0, r_0)$, then*

(a) *the sequence* $\{x_n\}$, $n = -1, 0, 1, 2, \cdots$ *is well defined on* $U = U(x_0, \mu_0)$, *where* $\mu_0 = \sigma(r_0)$ *remains in* U *and converges to a solution* x^* *of equation* (1) *which is unique in* $U(x_0, \mu_0)$, $\mu_0 < \mu$. *Moreover, the following estimates are true:*

$$\|x_{n+1} - x^*\| \leq \sigma(\omega^{(n)}(r_0)), \quad n = 0, 1, 2, \cdots \tag{34}$$

$$\|x_n - x^*\| \leq \sigma(\|x_n - x_{n-1}\|) - \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \cdots \tag{35}$$

where ω, σ *are given by* (31) *and* (32) *respectively.*

Proof. Let $A = \{u = (y, x) \in U^2;\ \delta f(y, x) \text{ is invertible}\}$ and $F : A \to E_1$ be a mapping such that

$$Fu = x - \delta f(y, x)^{-1} f(x). \tag{36}$$

Set $t_0 = (q_0, r_0)$. Then, by (31) and (32), $\sigma(t_0) = \mu_0$. Define the set

$$Z(t) = \{u = (y, x) \in E_1^2;\ y \in U, \|x - y\| \leq q, \|x - x_0\| \leq \mu_0 - \sigma(t),$$

$t = (q, r) \in T^2$, the linear operator $D = \delta f(y, x)$ is invertible and $\|D^{-1} f(x)\| \leq r\}$.

Then, we can immediately get $Z(t) \leq A$ and $u_0 = (x_{-1}, x_0) \in Z(t_0)$. We will now show that if $u = (y, x) \in Z(t)$, then $(x, Fu) \in Z(\omega(t))$. We denote $z = Fu$ and we will show that

$$x \in U, \ \|z - x\| \leq r, \tag{37}$$

$$\|z - x_0\| \leq \mu_0 - \sigma(\omega(t)), \tag{38}$$

the operator $D_1 = \delta f(x, z)$ is invertible and

$$\|D_1^{-1} f(z)\| \leq \omega(t). \tag{39}$$

Since $z - x = -D^{-1} f(x)$, estimate (37) follows at once. Then we have

$$\|z - x_0\| \leq \|z - x\| + \|x - x_0\| r + \mu_0 - \sigma(t) = \mu_0 - \sigma(\omega(t)),$$

since $\sigma(\omega(t)) = \sigma(t) - r$.

By (33) we have

$$\begin{aligned}\|D_0^{-1}(D_0 - D_1)\| &\le \|D_0^{-1}(D_0 - f'(x_0))\| + \|D_0^{-1}(f'(x_0) - D_1)\| \\ &\le h_0(\|x_0 - x_{-1}\|^q + \|x - x_0\|^q + \|z - x_0\|^q) \\ &\le h_0(2\mu_0^q + q_0^q) < 1, \quad \text{by (30)}.\end{aligned}$$

By the Banach lemma on invertible operators

$$\|D_1^{-1} D_0\| \le \frac{1}{1 - h_0(2\mu_0^q + q_0^q)} \tag{40}$$

Also, from (36) we obtain the approximation

$$f(x) \;=\; f(z) - f(x) - \delta f(y, x)(z - x),$$

and by (33) we get

$$\|D_0^{-1} f(z)\| \le h_0 \left(\frac{2}{1+q}\|z - x\|^q + \|x - y\|^q\right) \|z - x\|. \tag{41}$$

By (31), (40) and (41) we now obtain

$$\|D_1^{-1} f(z)\| \;=\; \|(D_0^{-1} D_1)^{-1} D_0^{-1} f(z)\| \le \omega(t).$$

The validity of (37), (38) and (39) has now been verified. It now follows that hypotheses (13), (14) and (15) of Proposition 1 are satisfied. Hence, the sequence (2) converges to a point $x^* \in A$ and estimates (34) are satisfied. Moreover, we have

$$\begin{aligned}&(x_{n-2}, x_{n-1}) \in Z(\omega^{(n-1)}(t_0)), \quad n \ge 1 \\ &\|x_{n+1} - x_n\| \le \omega^{(n)}(t_0), \quad n \ge 0.\end{aligned}$$

The function σ being increasing (in the sense that $q_1 \le q_2$ are $r_1 \le r_2$ implies $\sigma(q_1, r_1) \le \sigma(q_2, r_2)$, from the above, one can deduce that

$$(x_{n-2}, x_{n-1}) \in Z(\|x_{n-1} - x_{n-2}\|, \|x_n - x_{n-1}\|), \quad n \ge 1,$$

from which estimate (58) follows.

Set $t = \omega^{(n)}(t_0)$, $z = x_{n+1}$ in (41) and let n tend to infinity. Then $f(x^*) = 0$, which shows that x^* is a solution of equation (1). To show uniqueness, we assume that there exists another solution y^* of equation (1) in $U(x_0, \mu_0)$. Then using (26), we get

$$
\begin{aligned}
&\int_0^1 \|D_0^{-1}[f'(y^* + t(x^* - y^*)) - D_0]\| \, dt \\
&\leq \int_0^1 \|D_0^{-1}[f'(y^* + t(x^* - y^*)) - \delta f(x_0, x_0)]\| \, dt + \|D_0^{-1}(\delta f(x_0, x_0) - D_0)\| \\
&\leq h_0 \left[2 \int_0^1 ((1-t)^q \|y^* - x_0\|^q + t^q \|x^* - x_0\|^q) dt + q_0^q \right] \qquad (42) \\
&\leq h_0 \left(\frac{4}{q+1} \mu_0^q + q_0^q \right) < 1, \quad \text{by (30)}.
\end{aligned}
$$

Hence, the linear operator $\int_0^1 f'(y^* + t(x^* - y^*))dt$ is invertible. From the approximation

$$f(x^*) - f(y^*) = \int_0^1 f'(y^* + t(x^* - y^*))(x^* - y^*)dt,$$

it now follows that $x^* = y^*$, which completes the proof of the theorem.

Remark 1. The case $q = 1$ has been investigated already in the elegant work contained in [5] under different hypotheses. The main reason for our study is to cover cases when $q \in [0, 1)$, which are not covered in [5].

Remark 2. Let us consider the scalar equation

$$g(t) = 0$$

where

$$g(t) = \beta_0 t^{1+q} + \beta_1 t + \beta_2 t^q + \beta_3,$$

$$\beta_0 = 2h_0(1+q), \quad \beta_1 = -2h_0 r_0 (1+q),$$

$$\beta_2 = h_0(3+q) r_0^q - (1+q)(1 - h_0 q_0^q)$$

and

$$\beta_3 = r_0(1+q)(1-h_0 q_0^q).$$

Then, by (29) we have

$$g(\mu_0) = 0.$$

We can now provide sufficient conditions for the existence of two positive solutions of equation $g(t) = 0$.

Theorem 2. *Assume that the following conditions are satisfied:*

(a) $\beta_1 + \beta_2 < 0$; $g(0) > 0$;

(b) $(\beta_1 + \beta_2)^2 - 4\beta_0\beta_3 > 0$,

and

(c) $g\left(\frac{\beta_1 + \beta_2}{2\beta_0}\right) < 0$.

Then there exists a minimum positive solution μ_0 of equation

$$g(t) = 0.$$

Proof. The function $h(t) = \beta_0 t^2 + (\beta_1 + \beta_2)t + \beta_3$ has a minimum at $t_m = -\frac{\beta_1 + \beta_2}{2\beta_0}$. Hypotheses (a) and (b) imply that the equation $h(t) = 0$ has two positive solutions. By hypothesis (c), $g(t_m) < 0$. Since $g(t)$ is continuous, $g(0) > 0$ and $g(t) > 0$ for t sufficiently large, we conclude that $g(t)$ has two positive solutions μ_0 and μ_1 of which we choose the minimum on as our μ_0.

The proof of the theorem is now complete.

Remark 3. Under the hypotheses of Theorems 1 and 2, by using (42) the uniqueness of the solution x^* of equation (1) can be extended to the ball $\overline{U}(x_0, \mu_1)$. The existence of μ_1 is guaranteed by Theorem 2. To achieve this goal we just need to replace the existence condition (30) by

$$h_0\left[\frac{2}{q+1}(\mu_0^q + \mu_1^q) + q_0^q\right] < 1.$$

We can then also set $\mu = \mu_1$ in Theorem 1.

3. Applications

Example 1. Consider the function G defined on $[0, b]$ by

$$G(t) = \frac{2}{3}t^{\frac{3}{2}} + t - 3$$

for some $b > 0$.

Let $\| \ \|$ denote the max norm on $\mathbf{R}$, then

$$\|G''(t)\| = \max_{t\in[0,b]} \left|\frac{1}{2}t^{-\frac{1}{2}}\right| = \infty,$$

which implies that the basic hypothesis in [3] (or the Lipschitz continuity of f' for $q \neq 1$ in [5]) for the application of Newton's method is not satisfied for finding a solution of the equation

$$G(t) = 0. \tag{43}$$

However, it can easily be seen that $G'(t)$ is Hölder continuous on $[0, b]$ with

$$c = 1 \quad \text{and} \quad q = \frac{1}{2}.$$

Therefore, under the assumptions of Theorem 1, iteration (2) will converge to a solution t^* of equation (43).

Another interesting application for Theorem 1 is given by the following example.

Example 2. Consider the differential equation

$$x'' + x^{1+q} = 0, \quad q \in [0, 1] \tag{44}$$

$$x(0) = x(1) = 0.$$

We divide the interval $[0, 1]$ into n subintervals and we set $h = \frac{1}{n}$. Let $\{v_k\}$ be the points of subdivision with

$$0 = v_0 < v_1 \ldots < v_n = 1.$$

A standard approximation for the second derivative is given by

$$x_i'' = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2}; \quad x_i = x(v_i), \quad i = 1, 2, \cdots, n-1.$$

Take $x_0 = x_n = 0$ and define the operator $\tilde{F} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$\tilde{F}(x) = H(x) + h^2 \varphi(x) \tag{45}$$

$$H = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix},$$

$$\varphi(x) = \begin{bmatrix} x_1^{1+q} \\ x_2^{1+q} \\ \vdots \\ x_{n-1}^{1+q} \end{bmatrix},$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}.$$

Then

$$\tilde{F}'(x) = H + h^2(q+1) \begin{bmatrix} x_1^q & & & 0 \\ & x_2^q & & \\ & & \ddots & \\ 0 & & & x_{n-1}^q \end{bmatrix}.$$

Newton's method cannot be applied to the equation

$$\tilde{F}(x) = 0. \tag{46}$$

We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities of the form x_i^{-p} and they may not exist.

We will face the same difficulty in verifying the Lipschitz continuity of $\tilde{F}'$.

Let $x \in \mathbf{R}^{n-1}$, $H \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ and define the norms of x and H by

$$\|x\| = \max_{1 \le j \le n-1} |x_j|$$

$$\|H\| = \max_{1 \le j \le n-1} \sum_{k=1}^{n-1} |h_{jk}|.$$

For all $x, z \in \mathbf{R}^{n-1}$ for which $|x_i| > 0$, $|z_i| > 0$, $i = 1, 2, \cdots, n-1$ we obtain, for $q = \frac{1}{2}$ say,

$$\begin{aligned}\left\|\tilde{F}'(x) - \tilde{F}'(z)\right\| &= \left\|\text{diag}\left\{(1 + \frac{1}{2})h^2(x_j^{\frac{1}{2}} - z_j^{\frac{1}{2}})\right\}\right\| \\ &= \frac{3}{2}h^2 \max_{1 \le j \le n-1} \left|x_j^{\frac{1}{2}} - z_j^{\frac{1}{2}}\right| \le \frac{3}{2}h^2[\max|x_j - z_j|]^{\frac{1}{2}} \\ &= \frac{3}{2}h^2\|x - z\|^{\frac{1}{2}}.\end{aligned}$$

A linear operator $L \in L(\mathbf{R}^{n-1}, R^{n-1})$ can be represented by a matrix with entries q_{ij} and

$$\|L\| = \max\left\{\sum_{j=1}^{n-1} |q_{ij}|; 1 \le i \le n-1\right\}.$$

Let us denote by $\tilde{F}_1, \ldots, \tilde{F}_{n-1}$ the components of $\tilde{F}$. For each $v \in \mathbf{R}^{n-1}$ we can write

$$\tilde{F}(v) = \left(\tilde{F}_1(v), \ldots, \tilde{F}_{n-1}(v)\right)^{tr}.$$

Let $v, w \in \mathbf{R}^{n-1}$ and define $\delta\tilde{F}(v, w)$ by the matrix with entries

$$\begin{aligned}\delta\tilde{F}(v, w)_{ij} &= \frac{1}{v_j - w_j}(\tilde{F}_i(v_1, \ldots, v_j, w_{j+1}, \ldots, w_m) \\ &\quad - \tilde{F}_i(v_1, \ldots, v_{j-1}, w_j, \ldots, w_m)), m = n - 1. \qquad (47)\end{aligned}$$

It can easily be seen that the operator defined by (47) satisfies $\delta\tilde{F}(v, w) \in L(\mathbf{R}^{n-1}, \mathbf{R}^{n-1})$.

Denote by

$$P_j\tilde{F}_i(v) = \frac{\partial\tilde{F}_i(v)}{\partial v_j}, \quad i, j = 1, 2, \ldots, n - 1.$$

We can choose $n = 10$ which gives (9) equations for iteration (2), if we look at it as a system of linear equations given $z_{-1}, z_0 \in \mathbf{R}^9$. Since a solution would vanish at the end points and be positive in the interior a reasonable choice of initial approximation seems to be $130 \sin \pi x$. This gives us the following vector

$$z_{-1} = \begin{bmatrix} 4.015241\ E+01 \\ 7.637852\ E+01 \\ 1.051351\ E+02 \\ 1.236112\ E+02 \\ 1.299991\ E+02 \\ 1.236752\ E+02 \\ 1.052571\ E+02 \\ 7.654622\ E+01 \\ 4.034951\ E+01 \end{bmatrix}.$$

Choose z_0 by setting

$$z_0(v_i) = z_{-1}(v_i) - 10^{-5},\ i = 1, 2, \cdots, 9.$$

Using iteration (26) with the above values and (32), after seven iterations we get

$$z_6 = \begin{bmatrix} 3.357455\ E+01 \\ 6.520294\ E+01 \\ 9.156631\ E+01 \\ 1.091680\ E+02 \\ 1.153630\ E+02 \\ 1.091680\ E+02 \\ 9.156663\ E+02 \\ 6.520294\ E+01 \\ 3.357455\ E+01 \end{bmatrix} \text{ and } z_7 = \begin{bmatrix} 3.357450\ E+01 \\ 6.520290\ E+01 \\ 9.156660\ E+01 \\ 1.091680\ E+02 \\ 1.536301\ E+02 \\ 1.091680\ E+02 \\ 9.156660\ E+02 \\ 6.520290\ E+01 \\ 3.357450\ E+01 \end{bmatrix}.$$

We choose $z_6 = x_{-1}$ and $z_7 = x_0$ for our Theorem 1. From now on we assume that $\tilde{F}$ is restricted on $\overline{U}(x_0, \mu_0)$, $\mu_0 = 10^{-4}$. With the notation of Theorem 1 we can easily obtain the following results:

$$\|D_0^{-1}\| = 25.5882,$$
$$q = \frac{1}{2},$$

$$d = .03,$$
$$q_0 = 5.10^{-5},$$
$$\overline{r}_0 = r_0 = 9.15311.10^{-5},$$

and

$$h_0 = .767646.$$

All the hypotheses of Theorem 1 are now satisfied with the above values.

Therefore, the iteration generated by (2) converges to solution z^* of equation (46), which is unique in $\overline{U}(x_0, \mu_0)$.

Finally, note that the results in [3]-[9] cannot be applied here.

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