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A NOTE ON THE CONSISTENCY OF LIMITATION METHODS

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1.

We denote by w the space of all real or complex valued sequences. We consider matrix transformations $y_i = \sum_{j=0}^{\infty} a_{i,j}x_j$ (that is, $y = Ax$, where $x, y \in w$). We write $w_A = \{x : Ax \in w\}$. Let m, bv, c, c^0 be respectively the spaces of bounded, bounded variation, convergent and null sequences; $\ell = \{x : \sum_{n=0}^{\infty} |x_n| < \infty\}$; $bv^0 = bv \cap c^0$. We shall make use of the special sequences $e^k = (0,0,0,\cdots,1,0,0,0,\cdots)$, $e = (1,1,1,\cdots)$. Denote $\Delta = \{e^k\}_{k>0}$ and $\Delta^+ = \Delta + \{e\}$. We now define a sequence space of different type, namely

$$b_A = \{x \in w_A : \left|\sum_{k=m}^{\infty} a_{nk}x_k\right| \leq K(A,x) \quad (m,n = 0,1,2,\cdots)\}.$$

If $x \in c$ its limit is denoted by $\lim x$, and if $x \in c_A$ we denote $\lim(Ax) = \lim_A x$. Denote the set of all e's by

$$\phi = \{e, e^0, e^1, e^2, \cdots\}.$$

Then $e \in c_A$, $e^k \in c_A$ are respectively equivalent to the requirements that row sums of A exist and tend to a limit, or that the k-th column-limit of A exists. We denote $\chi_A = \lim_A e - \sum_k \lim_A e^k$ whenever this exists and is finite, and

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$\|A\| = \sup_n \sum_k |a_{nk}|$, finite or not. Note that $\ell_A \subseteq c_A^0 \subseteq c_A \subseteq m_A \subseteq w_A$ and $\ell_A \subseteq bv_A \subseteq c_A$.

We say that A defines a section bounded matrix transformation (introduced by Wilansky and Zeller [7]), when it belongs to the set

$$\mathcal{A} = \{A : c_A \subseteq b_A\}$$

Thus matrices in $\mathcal{A}$ have the property

$$\mathcal{A} : \forall x \in c_A, \sup_{m,n} \left| \sum_{k=m}^{\infty} a_{nk} x_k \right| = K(A, x) < \infty.$$

We define

$$\mathcal{A}^0 = \{A : c_A^0 \subseteq b_A\}.$$

A real matrix $H = (h_{nk})$ is called diagonal positive when

$$h_{nk} < 0 \ (0 \leq k < n), \ h_{nn} > 0, \ h_{nk} = 0 \ (k > n); \ n, k = 0, 1, 2, \cdots,$$

2.

In [2] the authors have obtained necessary and sufficient conditions for the consistency of two limitation methods for Nörlund summable sequences. The object of this note is to extend the results obtained in [2] and demonstrate that by using functional analysis technique the proof becomes easier.

In order to state our theorem we need some definitions.

Let $p = (p_n)$ and $q = (q_n)$ be sequences of real or complex numbers such that $q_n \neq 0$ for $n \geq 0$ and $q_n = 0$, $p_n = 0$, for $n < 0$. Let

$$r_n = \sum_{k=0}^{n} p_{n-k} q_k.$$

We assume that, for all n, $r_n \neq 0$. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with (s_n) as the sequence of its partial sums. Let (N, p, q) denotes the generalized Nörlund method in which the sequence (s_n) is transformed into τ_n given by

$$\tau_n = \frac{1}{r_n} \sum_{k=0}^{n} p_{n-k} q_k s_k \tag{1}$$

If $\tau_n \to s$ (finite) as $n \to \infty$, then the series $\sum_{n=0}^{\infty} a_n$ (or the sequence (s_n)) is said to be summable by the generalized Nörlund method (N, p, q) to s. We denote it by

$$\sum_{n=0}^{\infty} a_n = s(N, p, q) \quad \text{or} \quad s_n \to s(N, p, q).$$

The necessary and sufficient conditions for the regularity of (N, p, q) are

$$\sum_{v=0}^{n} |p_{n-v} q_v| = 0\,[|r_n|] \qquad (n \geq 0) \tag{i}$$

and

$$p_{n-v} = 0\,[|r_n|] \text{ as } n \to \infty \quad (v \text{ fixed}). \tag{ii}$$

This follows from Toeplitz's theorem (Hardy [1], Theorem 2). The method (N, p, q) reduces to the Nörlund method (N, p) when $q_n = 1$ for all n (Hardy [1], p.64) and to the Riesz method $(\overline{N}, q)$ when $p_n = 1$ for all n (Hardy [1], p.57).

Given any sequence (p_n), we write

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \quad p_{-m} > 0 \quad (m > 0)$$

and

$$[p(z)]^{-1} = \sum_{n=0}^{\infty} \gamma_n z^n$$

whenever the series on the right converges.

It is reasonably familiar that if we define

$$z_n = \sum_{k=0}^{n} p_{n-k} y_k$$

then

$$y_n = \sum_{k=0}^{n} \gamma_{n-k} z_k.$$

Applying this with $y_k = s_k q_k$, $z_k = r_k \tau_k$ we find that

$$s_n = \frac{1}{q_n} \sum_{k=0}^{n} \gamma_{n-k} \tau_k r_k.$$

As usual we say that the sequence $(p_n) \in \mathcal{M}$, if $p_0 = 1$,

$$p_n > 0, \ \frac{p_{n+1}}{p_n} \le \frac{p_{n+2}}{p_{n+1}} \le 1 \quad (n = 0, 1, \cdots).$$

Let A and B be two infinite matrices.

Let

$$u_m = \sum_{n=0}^{\infty} a_{mn} s_n, \quad v_m = \sum_{n=0}^{\infty} b_{mn} s_n.$$

We say that the A, B transforms are equiconvergent if

$$\sum_{n=0}^{\infty} (a_{mn} - b_{mn}) s_n \quad \text{converges for all } m,$$

and its sum tends to 0 as $m \to \infty$. We shall prove the following theorem.

Theorem 1. *Let (N,p) be a Nörlund method where $p_n \in \mathcal{M}$ and let (q_n) be a sequence where each $q_n \ge 1$. Then the necessary and sufficient conditions that A, B transforms are equiconvergent for all (N,p,q) summable sequences are*

(i) $\lim_{m\to\infty} d_{mn} = 0$ *for every fixed* n;

(ii) $\lim_{m\to\infty} \sum_{n=0}^{\infty} d_{mn} = 0$

(iii) $\sup_{m} \sum_{k=0}^{\infty} r_k \left| \sum_{n=k}^{\infty} \frac{d_{mn}}{q_n} \gamma_{n-k} \right| \le M$

where $d_{mn} = a_{mn} - b_{mn}$.

In order to prove the theorem we need some known results. We shall see that by using functional anaylysis technique the proof of Theorem 1 becomes easier.

Lemma 1. *If $p \in \mathcal{M}$ and if $q_n \geq \delta > 0$, $r_n = \sum_{k=0}^{n} p_{n-k}q_k$, then $H = (N, p, q)$ is non-negative, normal, regular, and has the normal inverse*

$$H^{-1} = (h_{nk}^{-1}), \; h_{nk}^{-1} = r_k \overline{p}_{n-k}/q_n \quad (0 \leq k < n),$$

where $\overline{p}_0 = 1/p_0$, $\overline{p}_k \leq 0$ $(k \geq 1)$; also $\sum_{k=0}^{n} h_{nk}^{-1} = 1$. It follows that H satisfies $h_{nk}^{-1} \leq 0$ $(0 \leq k \leq n)$, $\sum_{k=0}^{n} h_{nk}^{-1} \geq 0$, and hence H has the section bounded property.

This result is known and given in ([7] Kriterium 1, p. 260).

Theorem A. *Let H be normal, section bounded, and coregular. In order that $c_H \subseteq c_D$ it is necessary and sufficient that*

(i) *X_D exists;*

(ii) $\|DH^{-1}\| < \infty$

This theorem follows from a Theorem ([4], Theorem 3, p. 263).

Suppose that $\Delta = \{e^k\} \subseteq c_H^0$. This means that c_H^0 is an AK-space (Δ is a Schauder basis for c_H^0).

Theorem B. *Let H be normal, section-bounded, and $\Delta \subseteq c_H^0$. Then*

$$c_H^0 \subseteq c_D^0 \iff \begin{cases} \Delta \subseteq c_D^0 \\ T = DH^{-1} \text{ satisfies } D = TH, \; \|T\| < \infty \end{cases}$$

The proof of this theorem is similar to the proof given in ([4], Theorem 4). We shall now extend Theorem B. In order to extend Theorem B we need a very simple lemma.

Lemma 2. *If there is a sequence δ such that $H\delta = e$, then*

$$c_H \subseteq c_D^0 \iff \delta \in c_D^0 \text{ and } c_H^0 \subseteq c_D^0.$$

In the particular application $H = (N, p, q)$, we have $\sum_{k=0}^{n} h_{nk} = 1$ for every n, or that $He = e$ (i.e. $\delta = e$). So if we write $\Delta^+ = \Delta \cup \{e\} = \{e^k\} \cup \{e\}$, Theorem B and Lemma 2 yield:

Theorem C. *Let $H = (N, p, q)$ where $p \in \mathcal{M}$ and $q_n \geq \delta > 0$. Then*

$$c_H \subseteq c_D^0 \iff \begin{cases} \Delta^+ \subseteq c_D^0 \\ T = DH^{-1} \text{ satisfies } D = TH, \ \|T\| < \infty \end{cases}$$

Remark. More serious is the elimination of the criterion "$T = DH^{-1}$ satisfies $D = TH$" in Theorem C.

We shall now show that for the matrix $H = (N, p, q)$, this is true.

In order to see this we need the following results:

Lemma 3. *Let H be normal and have the inverse property. Let u be a positive sequence such that uH^{-1} exists and is non-negative. If d is a sequence such that*

$$\begin{aligned} d_k &= o(u_k) \text{ then } dH^{-1} \text{ and } (dH^{-1})H \text{ exist, and} \\ d &= (dH^{-1})H \end{aligned}$$

Proof. Note that H is non-negative. Since H^{-1} is diagonal positive and uH^{-1} exists, it follows that

$$\sum_{n=0}^{\infty} u_n \left|h_{nk}^{-1}\right| < \infty, \quad \text{and}$$

$d_n = o(u_n)$. This shows that dH^{-1} exists.

Choose any fixed $r \geq 0$; then, for any $N \geq r$

$$\begin{aligned} d_r &= \sum_{n=r}^{N} d_n \delta_{nr} \\ &= \sum_{n=r}^{N} d_n \sum_{k=r}^{n} h_{nk}^{-1} h_{kr} \\ &= \Big(\sum_{k=r}^{N} \sum_{n=k}^{\infty} - \sum_{n=r}^{N} \sum_{n=N+1}^{\infty}\Big) d_n h_{nk}^{-1} h_{nr} \\ &= S_1 - S_2 \text{ (since } dH^{-1} \text{ exists).} \end{aligned}$$

Write

$$\tau_N = \sup_{n>N+1}\left|\frac{d_n}{u_n}\right| = o(1) \text{ as } N\to\infty, \quad \text{since } \frac{d_n}{u_n} = o(1).$$

Then

$$|S_2| \le \sum_{k=r}^{N} h_{kr} \sum_{n=N+1}^{\infty} |d_n h_{nk}^{-1}|$$

$$\le \tau_N \sum_{k=r}^{N} h_{kr}\Big(-\sum_{n=N+1}^{\infty} u_n h_{nk}^{-1}\Big),$$

since $h_{nk}^{-1} \le 0$ for $n > k$,

$$\le \tau_N \sum_{k=r}^{N} h_{kr} \sum_{n=k}^{N} u_n h_{nk}^{-1},$$

since $(\sum_{n=k}^{N} + \sum_{n=N+1}^{\infty})u_n h_{nk}^{-1} \ge 0$,

$$= \tau_N \sum_{n=r}^{N} u_n \sum_{k=r}^{n} h_{nk}^{-1} h_{kr}$$

$$= \tau_N \sum_{n=r}^{N} u_n \delta_{nr} = \tau_N u_r \to 0 \text{ as } N \to \infty$$

It follows that $d_r = \lim_{N\to\infty} S_1 = [(dH^{-1})H]_r$ for $r = 0, 1, 2, \cdots$.

Lemma 4. *Let H be normal and have the inverse property. Let u be a positive sequence such that*

$$u_{k+1} h_{nk} \le u_k h_{n,k+1} \quad (0 \le k < n = 1, 2, \cdots).$$

Then uH^{-1} exists and is non-negative.

Proof. Write $T = (t_{nk})$, $t_{nk} = 1$ $(0 \le k \le n)$, $t_{nk} = 0$ $(k > n)$; its inverse is

$$T^{-1} = (t_{nk}^{-1}),\ t_{nn}^{-1} = 1,\ t_{nn-1}^{-1} = -1,\ t_{nk}^{-1} = 0 \quad \text{otherwise.}$$

Let

$$g_{nk} = \frac{h_{nk}}{u_n}; \text{ then } g_{nk}^{-1} = u_n g_{nk}^{-1} \text{ and, by hypothesis,}$$

$$(GT^{-1})_{nk} = g_{nk} - g_{n,k+1} \le 0 \text{ for } 0 \le k < n = 1,2,3,\cdots$$

Hence, $(GT^{-1})^{-1} = TG^{-1}$ is non-negative. That is,

$$0 \le (TG^{-1})_{NK} = \sum_{n=k}^{N} u_n h_{nk}^{-1} = u_k h_{nk}^{-1} - \sum_{n=k+1}^{N} u_n \left|h_{nk}^{-1}\right|,$$

and so $\sum_{n=k}^{\infty} u_n h_{nk}^{-1}$ converges and is non-negative, for each k.

In the case $H = (N, p, q)$ we have

$$h_{nk} = \frac{p_{n-k} q_k}{r_n} \quad (r_n = \sum_{k=0}^{n} p_{n-k} q_k)$$

Since $p \in \mathcal{M}$, it follows that

$$\frac{h_{nk}}{q_k} \le \frac{h_{n,k+1}}{q_{k+1}}$$

Hence, $u_k = q_k$ in the last two lemmas. This gives:

If $d_k = o(q_k)$, then $t = dH^{-1}$ exists and satisfies $d = tH$ where $H = (N, p, q)$. Moreover, since $q_k \ge \delta > 0$. It follows that

$$d_k = o(1) \Rightarrow d_k = o(q_k).$$

Replacing d_k by d_{nk} we have:

If $\lim_k d_{nk} = 0$ for each n, then $T = DH^{-1}$ exists and satisfies $D = TH$. Finally, the condition $e \in c_D^o$ (included in the condition $\Delta^+ \subseteq c_D^o$ of Theorem C) says that

$$\lim_n \sum_{k=0}^{\infty} d_{n,k} = 0, \quad \text{and in particular this implies that}$$

$$\lim_k d_{nk} = 0 \quad \text{for each } n.$$

Thus Theorem C reduces to the form

Theorem C′ *Let $H = (N, p, q)$, where $p \in \mathcal{M}$ and $q_n \geq \delta > 0$. Then*

$$c_H \subseteq c_D^o \iff \Delta^+ \subseteq c_D^o \text{ and } \|DH^{-1}\| < \infty$$

The conditions on the right of Theorem C′ are:

(i) $\lim_m d_{mn} = 0$ *for each* $n(e^n \in c_D^o)$.

(ii) $\lim_m \sum_{n=0}^{\infty} d_{mn} = 0 \ (e \in c_D^o)$

(iii) $\sup_m \sum_{k=0}^{\infty} r_k \left| \sum_{n=k}^{\infty} \frac{d_{mn}}{q_n} \gamma_{n-k} \right| < M \ (\|DH^{-1}\| < \infty)$.

Thus the proof of Theorem 1 is complete.

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