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INVARIANT APPROXIMATIONS, GENERALIZED $\mathscr{I}$-NONEXPANSIVE MAPPINGS AND NON-CONVEX DOMAIN

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Abstract. Common fixed point results for generalized $\mathscr{I}$-nonexpansive mappings, and nonlinear map on nonstarshaped domain have been obtained in the present work. Various invariant approximation results have also been determined by its application. These results extend and generalize various existing known results in the literature. A property called property (Γ) has also been introduced in it.

1. Introduction

Fixed point theorems have been applied in the field of invariant approximation theory since last four decades and several interesting and valuable results have been studied.

Meinardus [9] was the first to employ a fixed-point theorem of Schauder to establish the existence of an invariant approximation. Further, Brosowski [3] obtained a celebrated result and generalized the Meinardus's result. Later, several results [7, 14, 16] have been proved in the direction of Brosowski [3]. In the year 1988, Sahab, Khan and Sessa [11] extended the result of Hicks and Humpheries [7] and Singh [14] by considering one linear and the other nonexpansive mappings.

Al-Thagafi [1] generalized result of Sahab, Khan and Sessa [11] and proved some results on invariant approximations for commuting mappings. The introduction of non-commuting maps to this area, Shahzad [12, 13] further extended Al-Thagafi's results and obtained a number of results regarding invariant approximation. All the above mentioned results are obtained on starshaped domain and linearity or affinness condition of mapping.

In this context, it may be mentioned that Dotson [4] proved the existence of fixed point for nonexpansive mapping. He further extended his result without starshapedness under non-convex condition [5]. Mukherjee and Som [10] used it to prove existence of fixed point and further applied it for proving existence of best approximant. This resulted in extension of Singh [14] without starshapedness condition.

Attempt has been made to find existence results on common fixed point theorem to generalize $\mathscr{I}$-nonexpansive maps non-linear map to a domain which is not necessarily starshaped.

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Various invariant approximation results have also been obtained. These results extend, generalize, and compliment those of Al-Thagafi [1], Dotson [4, 5], Habiniak [6], Mukherjee and Som [10], Sahab, Khan and Sessa [11], Shahzad [12, 13] and Singh [14]. For it, a property called property (Γ) has been introduced.

2. Preliminaries

In the material to be produced have, the following definitions have been used:

Definition 2.1.([1]) Let $\mathscr{X}$ be a normed space and let $\mathscr{C}$ be a non-empty subset of $\mathscr{X}$. Let $x_0 \in \mathscr{X}$. An element $y \in \mathscr{C}$ is called a best approximant to $x_0 \in \mathscr{X}$, if

$$\|x_0 - y\| = dist(x_0, \mathscr{C}) = \inf\{\|x_0 - z\| : z \in \mathscr{C}\}.$$

Let $\mathscr{P}_{\mathscr{C}}(x_0)$ be the set of best $\mathscr{C}$-approximant to x_0 and so

$$\mathscr{P}_{\mathscr{C}}(x_0) = \{z \in \mathscr{C} : \|x_0 - z\| = dist(x_0, \mathscr{C})\}.$$

Definition 2.2.([1]) Let $\mathscr{X}$ be a normed space. A set $\mathscr{C}$ in $\mathscr{X}$ is said to be convex, if $\lambda x + (1-\lambda) y \in \mathscr{C}$, whenever $x, y \in \mathscr{C}$ and $0 \le \lambda \le 1$.

A set $\mathscr{C}$ in $\mathscr{X}$ is said to be starshaped, if there exists at least one point $p \in \mathscr{C}$ such that the line segment $[x, p]$ joining x to p is contained in $\mathscr{C}$ for all $x \in \mathscr{C}$ (that is $\lambda x + (1-\lambda) p \in \mathscr{C}$, for all $x \in \mathscr{C}$ and $0 < \lambda < 1$). In this case p is called a starcenter of $\mathscr{C}$.

Each convex set is starshaped with respect to each of its points, but not conversely.

Definition 2.3.([11]) A map $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ is said to be $\mathscr{I}-$contraction, if there exists a self-map $\mathscr{I}$ on $\mathscr{C}$ and a real number $k \in (0,1)$ such that

$$\|\mathscr{T} x - \mathscr{T} y\| \le k \|\mathscr{I} x - \mathscr{I} y\|,$$

for all $x, y \in \mathscr{C}$. If $k = 1$, then $\mathscr{T}$ is called $\mathscr{I}-$nonexpansive.

Definition 2.4. The map $\mathscr{T} : \mathscr{C} \to \mathscr{X}$ ($\mathscr{C}$ is subset of $\mathscr{X}$) is said to be completely continuous if $\{x_n\}$ converges weakly to x implies that $\{\mathscr{T} x_n\}$ converges strongly to $\mathscr{T} x$.

Definition 2.5.([13]) A pair $(\mathscr{I}, \mathscr{T})$ of self-mappings of a Banach space $\mathscr{X}$ is said to be

(1) commutative on $\mathscr{C}$ ($\mathscr{C} \subset \mathscr{X}$), if $\mathscr{I}\mathscr{T} x = \mathscr{T}\mathscr{I} x$ for all $x \in \mathscr{C}$;

(2) $\mathscr{R}$-weakly commuting on $\mathscr{C}$, it there exists a real number $\mathscr{R} > 0$ such that $\|\mathscr{T}\mathscr{I} x - \mathscr{I}\mathscr{T} x\| \le \mathscr{R} \|\mathscr{T} x - \mathscr{I} x\|$ for all $x \in \mathscr{C}$.

Suppose $\mathscr{C}$ is $p-$starshaped with $p \in F(\mathscr{I})$ and is both $\mathscr{T}-$ and $\mathscr{I}-$ invariant. Then $\mathscr{T}$ and $\mathscr{I}$ are called

(3) $\mathscr{R}$-subweakly commuting on $\mathscr{C}$ [13] if there exists $\mathscr{R} \in (0, \infty)$ such that $\|\mathscr{T}\mathscr{I} x - \mathscr{I}\mathscr{T} x\| \le \mathscr{R}\ dist(\mathscr{I} x, [\mathscr{T} x, p])$ for all $x \in \mathscr{C}$, where $dist(\mathscr{I} x, [\mathscr{T} x, p]) = \inf\{\|\mathscr{I} x - z\| : z \in [\mathscr{T} x, p]\}$. Clearly commutativity implies $\mathscr{R}$-subweak commutativity, but the converse may not be true [13].

For this, the following example is considered:

Example 2.6.([12]) Consider $\mathscr{X} = \mathbb{R}$ with norm $\|x\| = |x|$ and $\mathscr{Y} = [1, \infty)$. Let $\mathscr{T}$ and $\mathscr{I}$ be defined by

$$\mathscr{T}x = 4x - 3, \ \mathscr{I}x = 2x^2 - 1$$

for all $x \in \mathscr{X}$. Then $\mathscr{T}$ and $\mathscr{I}$ are $\mathscr{R}$-subweakly commuting on $\mathscr{Y}$. However, they are not commuting on $\mathscr{Y}$.

Further, definition providing the notion of contractive jointly continuous family introduced by Dotson [5] may be written as:

Definition 2.7.([5]) Let $\mathscr{C}$ be a subset of metric space $(\mathscr{X}, d)$ and $\mathscr{F} = \{f_\alpha\}_{\alpha \in \mathscr{C}}$ a family of functions from $[0,1]$ into $\mathscr{C}$ such that $f_\alpha(1) = \alpha$ for each $\alpha \in \mathscr{C}$. The family $\mathscr{F}$ is said to be *contractive* if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in \mathscr{C}$ and all $t \in (0,1)$ we have

$$d(f_\alpha(t), f_\beta(t)) \le \phi(t) d(\alpha, \beta).$$

The family is said to be *jointly continuous* if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in $\mathscr{C}$ imply that $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in $\mathscr{X}$.

Definition 2.8.([5]) If $\mathscr{X}$ is a normed linear space and $\mathscr{F}$ is a family as in Definition 2.7, then $\mathscr{F}$ is said to be *jointly weakly continuous* if $t \to t_0$ in $[0,1]$ and $\alpha \to^w \alpha_0$ in $\mathscr{C}$ imply that $f_\alpha(t) \to^w f_{\alpha_0}(t_0))$ in $\mathscr{X}$.

Hence, *property* (Γ) on contractive jointly continuous family $\mathscr{F}$ can now be defined as:

Definition 2.9. A self mapping $\mathscr{I}$ of $\mathscr{C}$ is said to satisfy the property (Γ), if for any $t \in [0,1]$, for all $x \in \mathscr{C}$ and for all $f_x \in \mathscr{F}$, we have $\mathscr{I}(f_x(t)) = f_{\mathscr{I}x}(t)$, where $\{f_x(t)\}$ is defined as above.

For clarification of a metric space that satisfies the notion of a contractive and jointly continuous family of functions, a lemma is presented below, it gives the concept of contractive and jointly continuous family of functions. It also implies that in Euclidean n-space such a set must be connected.

Lemma 2.10. *Let $(\mathscr{X}, d)$ be a metric space and $\mathscr{C}$ a nonempty subset which (as a subspace) is not connected. Suppose that $\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1$, $\mathscr{C}_0 \cap \mathscr{C}_1 = \phi$ where $\mathscr{C}_0$ and $\mathscr{C}_1$ are both open and closed, and suppose that there exist $x \in \mathscr{C}_0$ and $y \in \mathscr{C}_1$ such that $d(x, y) = d(\mathscr{C}_0, \mathscr{C}_1)$. Then $\mathscr{C}$ does not admit a jointly continuous contractive family $\mathscr{F} = \{f_\alpha\}_{\alpha \in \mathscr{C}}$; i.e. $\mathscr{C}$ does not have the property of contractiveness and joint continuity.*

Proof. We first note that the distance between $\mathscr{C}_0$ and $\mathscr{C}_1$ must be positive since otherwise we would have $x = y$, contradicting $\mathscr{C}_0 \cap \mathscr{C}_1 = \phi$. If $\mathscr{C}$ had a jointly continuous contractive family $\mathscr{F} = \{f_\alpha\}_{\alpha \in \mathscr{C}}$, then, since $d(\mathscr{C}_0, \mathscr{C}_1) > 0$, by taking t sufficiently close to 1 we shall have $d(x, f_x(t)) < d(\mathscr{C}_0, \mathscr{C}_1)$ and $d(y, f_y(t)) < d(\mathscr{C}_0, \mathscr{C}_1)$ by joint continuity, so that we must have $f_x(t) \in \mathscr{C}_0$ and $f_y(t) \in \mathscr{C}_1$. This leads to

$$d(\mathscr{C}_0, \mathscr{C}_1) \le d(f_x(t), f_y(t)) \le \phi(t) d(x, y) < d(x, y) = d(\mathscr{C}_0, \mathscr{C}_1)$$

a contradiction.

A consequence of this lemma is that, in a finite-dimensional Banach space, every bounded subset (considered as a metric space) that has the property of contractiveness and joint continuity must be connected. For closed bounded sets are compact, and the conditions of the lemma are satisfied in this case.

Common fixed point result given below is a consequence of Theorem 1 of Berinde [2], and would be needed in the sequel.

Theorem 2.11.([13, Theorem 2.1]) *Let $\mathscr{C}$ be a closed subset of a metric space $(\mathscr{X},d)$ and $\mathscr{T}$ and $\mathscr{I}$ be $\mathscr{R}$-weakly commuting self-maps of $\mathscr{C}$ such that $\mathscr{T}(\mathscr{C}) \subset \mathscr{I}(\mathscr{C})$. Suppose there exists $\lambda \in (0,1)$ such that*

$$d(\mathscr{T}x,\mathscr{T}y) \leq \lambda \max\{d(\mathscr{I}x,\mathscr{I}y), d(\mathscr{T}x,\mathscr{I}x), d(\mathscr{T}y,\mathscr{I}y), d(\mathscr{T}x,\mathscr{I}y), d(\mathscr{T}y,\mathscr{I}x)]\}$$

for all $x, y \in \mathscr{C}$. If $cl(\mathscr{T}(\mathscr{C}))$ is complete and $\mathscr{T}$ is continuous, then there is a unique point z in $\mathscr{C}$ such that $\mathscr{T}z = \mathscr{I}z = z$.

$F(\mathscr{T})$(resp. $F(\mathscr{I})$) has been denoted as the set of fixed point of mapping $\mathscr{T}$ (resp. $\mathscr{I}$) in the following text.

3. Main results

One may now prove common fixed theorem in nonconvex domain.

Theorem 3.1. *Let $\mathscr{C}$ be a closed subset of a metric space $(\mathscr{X},d)$ and $\mathscr{T}$ and $\mathscr{I}$ continuous self-mappings of $\mathscr{C}$ such that $\mathscr{T}(\mathscr{C}) \subset \mathscr{I}(\mathscr{C})$. Suppose that $\mathscr{C}$ admits a contractive and jointly continuous family $\mathscr{F} = \{f_\alpha\}_{\alpha\in\mathscr{C}}$, and that $\mathscr{I}$ satisfies property (Γ). Suppose further that $\mathscr{T}$ and $\mathscr{I}$ satisfy the following conditions:*

$$d(\mathscr{T}\mathscr{I}x, \mathscr{I}\mathscr{T}x) \leq \mathscr{R} d(f_{\mathscr{T}x}(k), \mathscr{I}x) \tag{3.1}$$

$$\begin{aligned} d(\mathscr{T}x,\mathscr{T}y) \leq \max\{&d(\mathscr{I}x,\mathscr{I}y), d(f_{\mathscr{T}x}(k),\mathscr{I}x), d(f_{\mathscr{T}y}(k),\mathscr{I}y),\\ &d(f_{\mathscr{T}y}(k),\mathscr{I}x), d(f_{\mathscr{T}x}(k),\mathscr{I}y)\} \end{aligned} \tag{3.2}$$

for all $x, y \in \mathscr{C}$, all $k \in (0,1)$ and some $\mathscr{R} > 0$. If $cl(\mathscr{T}(\mathscr{C}))$ is compact, then $\mathscr{C} \cap F(\mathscr{T}) \cap F(\mathscr{I}) \neq \phi$.

Proof. Choose a sequence $k_n \in (0,1)$ with $k_n \to 1$ as $n \to \infty$, and define, for each $n \in \mathbb{N}$, the mapping

$$\mathscr{T}_n x = f_{\mathscr{T}x}(k_n).$$

Each $\mathscr{T}_n$ maps $\mathscr{C}$ into itself, and $\mathscr{T}_n(\mathscr{C}) \subset \mathscr{I}(\mathscr{C})$. Indeed, let $y \in \mathscr{C}$. Since $\mathscr{T}(\mathscr{C}) \subset \mathscr{I}(\mathscr{C})$, we have $\mathscr{T}y = \mathscr{I}z$ for some $z \in \mathscr{C}$. Then, by property (Γ)

$$\mathscr{T}_n y = f_{\mathscr{T}y}(k_n) = f_{\mathscr{I}z}(k_n) = \mathscr{I}f_z(k_n) \in \mathscr{I}(\mathscr{C}).$$

Again by property (Γ), we have

$$\begin{aligned} d(\mathscr{T}_n\mathscr{I}x,\mathscr{I}\mathscr{T}_n x) &\le d(f_{\mathscr{T}\mathscr{I}x}(k_n),\mathscr{I}f_{\mathscr{T}x}(k_n)) = d(f_{\mathscr{T}\mathscr{I}x}(k_n),f_{\mathscr{I}\mathscr{T}x}(k_n)) \\ &\le \phi(k_n)d(\mathscr{T}\mathscr{I}x,\mathscr{I}\mathscr{T}x) \le \phi(k_n)\mathscr{R},d(f_{\mathscr{T}x}(k_n),\mathscr{I}x) \\ &= \phi(k_n)\mathscr{R},d(\mathscr{T}_n x,\mathscr{I}x) \end{aligned}$$

using (3.1). Thus $\mathscr{T}_n$ and $\mathscr{I}$ are $\phi(k_n)\mathscr{R}-$ weakly commuting. By contractiveness and (3.2), we have

$$\begin{aligned} d(\mathscr{T}_n x,\mathscr{T}_n y) &= d(f_{\mathscr{T}x}(k_n),f_{\mathscr{T}y}(k_n)) \le \phi(k_n)d(\mathscr{T}x,\mathscr{T}y) \\ &\le \phi(k_n)\max\{d(\mathscr{I}x,\mathscr{I}y),d(f_{\mathscr{T}x}(k_n),\mathscr{I}x),d(f_{\mathscr{T}y}(k_n),\mathscr{I}y), \\ &\quad d(f_{\mathscr{T}y}(k_n),\mathscr{I}x),d(f_{\mathscr{T}x}(k_n),\mathscr{I}y)\} \\ &\le \phi(k_n)\max d(\mathscr{I}x,\mathscr{I}y),d(\mathscr{T}_n x,\mathscr{I}x),d(\mathscr{T}_n y,\mathscr{I}y),d(\mathscr{T}_n y,\mathscr{I}x),d(\mathscr{T}_n x,\mathscr{I}y)\}. \end{aligned}$$

Since $cl(\mathscr{T}(\mathscr{C}))$ is compact, $cl(\mathscr{T}_n(\mathscr{C}))$ is also compact. Indeed, if (x_m) is a sequence in $\mathscr{T}_n(\mathscr{C})$ then $x_m = \mathscr{T}_n y_m$ for some y_m. Now some subsequence of $(\mathscr{T}y_m)$ converges since $cl(\mathscr{T}(\mathscr{C}))$ is compact, and since $x_m = f_{\mathscr{T}y_m}(k_n)$, that same subsequence of (x_m) converges to a point in $cl(\mathscr{T}_n(\mathscr{C}))$. Hence $cl(\mathscr{T}_n(\mathscr{C}))$ is compact. Since $\phi(k_n) < 1$, Theorem 2.11 now yield $x_n \in \mathscr{C}$ such that $x_n = \mathscr{T}_n x_n = \mathscr{I}x_n$. Since $cl(\mathscr{T}(\mathscr{C}))$ is compact, $(\mathscr{T}x_n)$ contains a convergent subsequence, say, $\mathscr{T}x_m \to y \in cl(\mathscr{T}(\mathscr{C}))$. Then we have

$$x_m = \mathscr{T}_m x_m = f_{\mathscr{T}x_m}(k_m) \to f_y(1) = y.$$

Since $\mathscr{T}$ is continuous, $\mathscr{T}x_m \to \mathscr{T}y$ and since $\mathscr{T}x_m \to y$, we have $y = \mathscr{T}y$. From the continuity of $\mathscr{I}$ we have $x_m = \mathscr{I}x_m \to \mathscr{I}y$, so that $\mathscr{I}y = y$ and we have $\mathscr{C}\cap F(\mathscr{T})\cap F(\mathscr{I}) \ne \phi$.

Theorem 3.2. *Let $\mathscr{C}$ be a subset of a Banach space $\mathscr{X}$ and $\mathscr{T}$ and $\mathscr{I}$ self-mappings of $\mathscr{C}$ such that $\mathscr{T}(\mathscr{C}) \subset \mathscr{I}(\mathscr{C})$. Suppose that $\mathscr{C}$ has a contractive family of functions $\mathscr{F} = \{f_\alpha\}_{\alpha\in\mathscr{C}}$, that $\mathscr{I}$ satisfies property* (Γ) *and that $\mathscr{T}$ and $\mathscr{I}$ satisfy* (3.1) *and* (3.2) *of Theorem* 3.1 *(d is the metric induced on $\mathscr{C}$ from $\mathscr{X}$). If $\mathscr{T}$ is continuous, then $\mathscr{C}\cap F(\mathscr{T})\cap F(\mathscr{I}) \ne \phi$, provided one of the following conditions holds:*

(i) *$\mathscr{C}$ is weakly compact, $\mathscr{T}$ and $\mathscr{I}$ are weakly continuous and the family $\mathscr{F}$ is weakly jointly continuous.*

(ii) *$\mathscr{C}$ is weakly compact, $\mathscr{T}$ is completely continuous, $\mathscr{I}$ is continuous, and the family $\mathscr{F}$ is jointly continuous.*

Proof. (i) As in Theorem 3.1, there exists $x_n \in \mathscr{C}$ such that $x_n = \mathscr{T}_n x_n = \mathscr{I}x_n$. Since $\mathscr{C}$ is weakly compact, (x_n) contains a convergent subsequence, say, (x_m) such that $x_m \to u \in \mathscr{C}$. Since $\mathscr{T}$ is weakly continuous, $\mathscr{T}x_m \to^w \mathscr{T}u$ and hence $x_m = f_{\mathscr{T}x_m}(k_m) \to f_{\mathscr{T}u}(1) = \mathscr{T}u$. Also since $x_m \to u$ and the weak topology is Hausdorff, we have $\mathscr{T}u = u$. From the weakly continuity of $\mathscr{I}$ we have $x_m = \mathscr{I}x_m \to \mathscr{I}u$, so that $\mathscr{I}u = u$. Hence $\mathscr{C}\cap F(\mathscr{T})\cap F(\mathscr{I}) \ne \phi$.

(ii) As in Theorem 3.1, there exists $x_n \in \mathscr{C}$ such that $x_n = \mathscr{T}_n x_n = \mathscr{I}x_n$. Since $\mathscr{C}$ is weakly compact, (x_n) contains a convergent subsequence, say, (x_m) such that $x_m \to u \in \mathscr{C}$. Since $\mathscr{T}$ is completely continuous, $\mathscr{T}x_m \to \mathscr{T}y$ as $m \to \infty$. Then we have

$$x_m = f_{\mathscr{T}x_m}(k_m) \to f_{\mathscr{T}y}(1) = \mathscr{T}y.$$

Thus $\mathcal{T}x_m \to \mathcal{T}^2 y$ and consequently $\mathcal{T}^2 y = \mathcal{T}y$ implies that $\mathcal{T}z = z$, where $z = \mathcal{T}y$. But, since $\mathcal{I}x_m = x_m \to \mathcal{T}y = z$, using the continuity of $\mathcal{I}$ and the uniqueness of the limit, we have $\mathcal{I}z = z$. Hence $\mathcal{C} \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

From Theorem 3.1, one obtains the following:

Corollary 3.3. *Let $\mathcal{C}$ be a closed subset of a metric space $(\mathcal{X}, d)$ and $\mathcal{T}$ and $\mathcal{I}$ continuous self-mappings of $\mathcal{C}$ such that $\mathcal{T}(\mathcal{C}) \subset \mathcal{I}(\mathcal{C})$. Suppose that $\mathcal{C}$ admits a contractive and jointly continuous family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{C}}$, and that $\mathcal{I}$ satisfies property (Γ). Suppose further that $\mathcal{T}$ and $\mathcal{I}$ satisfy (3.1) and $\mathcal{T}$ is $\mathcal{I}$-nonexpansive on $\mathcal{C}$. If $cl(\mathcal{T}(\mathcal{C}))$ is compact, then $\mathcal{C} \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.*

Corollary 3.4. *Let $\mathcal{C}$ be a closed subset of a metric space $(\mathcal{X}, d)$ and $\mathcal{T}$ and $\mathcal{I}$ continuous self-mappings of $\mathcal{C}$ such that $\mathcal{T}(\mathcal{C}) \subset \mathcal{I}(\mathcal{C})$. Suppose that $\mathcal{C}$ admits a contractive and jointly continuous family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{C}}$. Suppose further that $\mathcal{T}$ and $\mathcal{I}$ are commutative and satisfy (3.2) for all $x, y \in \mathcal{C}$. If $cl(\mathcal{T}(\mathcal{C}))$ is compact, then $\mathcal{C} \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.*

Remark 3.5. In the light of the comment given by Dotson [5] and Khan, Latif, Bano and Hussain [8] if $\mathcal{C} \subseteq \mathcal{X}$ is p−starshaped and $f_\alpha(t) = (1-t)p + t\alpha$, $(\alpha \in \mathcal{C}, t \in [0,1])$, then $\{f_\alpha\}_{\alpha\in\mathcal{C}}$ is a contractive jointly continuous family with $\phi(t) = t$. Thus the class of subsets of $\mathcal{X}$ with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contain the class of convex sets. If for a subset $\mathcal{C}$ of $\mathcal{X}$, there exists a contractive jointly continuous of family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{C}}$, then we say that $\mathcal{C}$ has the property of contractiveness and joint continuity.

Corollary 3.6. *Let $\mathcal{C}$ be a closed subset of a metric space $(\mathcal{X}, d)$ and $\mathcal{T}$ and $\mathcal{I}$ continuous self-mappings of $\mathcal{C}$ such that $\mathcal{T}(\mathcal{C}) \subset \mathcal{I}(\mathcal{C})$. Suppose $\mathcal{C}$ is q-starshaped, and $\mathcal{I}$ is affine with respect to $q \in F(\mathcal{I})$. Suppose further that $\mathcal{T}$ and $\mathcal{I}$ are $\mathcal{R}$-subweakly commuting and satisfy, for all $x, y \in \mathcal{C}$,*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \leq \max\{& d(\mathcal{I}x, \mathcal{I}y), dist([\mathcal{T}x, q], \mathcal{I}x), dist([\mathcal{T}y, q], \mathcal{I}y), \\ & \frac{1}{2}[dist([\mathcal{T}y, q], \mathcal{I}x) + dist([\mathcal{T}x, q], \mathcal{I}y)]\}. \end{aligned} \tag{3.3}$$

If $cl(\mathcal{T}(\mathcal{C}))$ is compact, then $\mathcal{C} \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem 3.1.

Remark 3.7. One can obtain similar Corollary 3.3, 3.4 and 3.6 from Theorem 3.2.

As an application of Theorem 3.1, we have following result on invariant approximations:

Following Al-Thafgafi [1], one defines $\mathcal{D} = \mathcal{P}_{\mathcal{C}}(x_0) \cap \mathcal{D}_{\mathcal{C}}^{\mathcal{I}}(x_0)$, where $\mathcal{D}_{\mathcal{C}}^{\mathcal{I}}(x_0) = \{x \in \mathcal{C} : \mathcal{I}x \in \mathcal{P}_{\mathcal{C}}(x_0)\}$.

Theorem 3.8. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of $\mathcal{X}$. Let $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Suppose that $\mathcal{P}_{\mathcal{C}}(x_0)$ is nonempty, $\mathcal{D}$ has a contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{D}}$, $\mathcal{I}$ satisfies the property (Γ) on $\mathcal{D}$, $\mathcal{I}(\mathcal{D}) = \mathcal{D}$ and $\mathcal{I}$ is* *nonexpansive on* $\mathscr{P}_{\mathscr{C}}(x_0)\cup\{x_0\}$. *Suppose further that* $\mathscr{T}$ *and* $\mathscr{I}$ *satisfy* (3.1) *for all* $x\in\mathscr{D}$, *some* $\mathscr{R}>0$, *and satisfy*

$$\|\mathscr{T}x-\mathscr{I}x\|\leq\begin{cases}\|\mathscr{I}x-\mathscr{I}x_0\|, & \text{if } y=x_0;\\ \max\{\|\mathscr{I}x-\mathscr{I}y\|,\|\mathscr{I}x-f_{\mathscr{T}x}(k)\|,\|\mathscr{I}y-f_{\mathscr{T}x}(k)\|, & \\ \|\mathscr{I}y-f_{\mathscr{T}x}(k)\|,\|\mathscr{I}y-f_{\mathscr{T}x}(k)\|\}, & \text{if } y\in\mathscr{D}.\end{cases}\tag{3.4}$$

for all $x,y\in\mathscr{D}\cup\{x_0\}$ *and all* $k\in(0,1)$. *If* $\mathscr{T}$ *is continuous, then* $\mathscr{P}_{\mathscr{C}}(x_0)\cap F(\mathscr{T})\cap F(\mathscr{I})\neq\phi$, *provided one of the following conditions holds:*

(i) $\mathscr{D}$ *is closed,* $cl(\mathscr{T}(\mathscr{D}))$ *is compact,* $\mathscr{I}$ *is continuous, and family* $\mathscr{F}$ *is jointly continuous,*

(ii) $\mathscr{X}$ *is Banach space,* $\mathscr{D}$ *is weakly compact,* $\mathscr{I}$ *and* $\mathscr{T}$ *are weakly continuous, family* $\mathscr{F}$ *is weakly jointly continuous,*

(iii) $\mathscr{X}$ *is Banach space,* $\mathscr{D}$ *is weakly compact,* $\mathscr{T}$ *is completely continuous,* $\mathscr{I}$ *is continuous, and family* $\mathscr{F}$ *is jointly continuous.*

Proof. Let $x\in\mathscr{D}$. Then, $x\in\mathscr{P}_{\mathscr{C}}(x_0)$ and hence $\|x-x_0\|=dist(x_0,\mathscr{C})$. Note that for any $k\in(0,1)$,

$$\|kx_0+(1-k)x-x_0\|=(1-k)\|x-x_0\|<dist(x_0,\mathscr{C}).$$

It follows that the line segment $\{kx_0+(1-k)x:0<k<1\}$ and the set $\mathscr{C}$ are disjoint. Thus x is not in the interior of $\mathscr{C}$ and so $x\in\partial\mathscr{C}\cap\mathscr{C}$. Since $\mathscr{T}(\partial\mathscr{C}\cap\mathscr{C})\subset\mathscr{C}$, $\mathscr{T}x$ must be in $\mathscr{C}$. Also since $\mathscr{I}x\in\mathscr{P}_{\mathscr{C}}(x_0)$, $x_0=\mathscr{T}x_0=\mathscr{I}x_0$ and $\mathscr{T}$ and $\mathscr{I}$ satisfy (3.4), we have

$$\|\mathscr{T}x-x_0\|=\|\mathscr{T}x-\mathscr{T}x_0\|\leq\|\mathscr{I}x-\mathscr{I}x_0\|=\|\mathscr{I}x-x_0\|=dist(x_0,\mathscr{C}).$$

Thus, $\mathscr{T}x\in\mathscr{P}_{\mathscr{C}}(x_0)$. As $\mathscr{I}$ is nonexpansive on $\mathscr{P}_{\mathscr{C}}(x_0)\cup\{x_0\}$, we have

$$\|\mathscr{I}\mathscr{T}x-x_0\|\leq\|\mathscr{T}x-\mathscr{T}x_0\|\leq\|\mathscr{I}x-\mathscr{I}x_0\|=\|\mathscr{I}x-x_0\|=dist(x_0,\mathscr{C}).$$

Thus $\mathscr{I}\mathscr{T}x\in\mathscr{P}_{\mathscr{C}}(x_0)$ and so $\mathscr{T}x\in\mathscr{D}^{\mathscr{I}}_{\mathscr{C}}(x_0)$. Hence $\mathscr{T}x\in\mathscr{D}$. Consequently, $\mathscr{T}(\mathscr{D})\subset\mathscr{D}=\mathscr{I}(\mathscr{D})$. Now Theorem 3.1 and 3.2 guarantee that

$$\mathscr{P}_{\mathscr{C}}(x_0)\cap F(\mathscr{T})\cap F(\mathscr{I})\neq\phi.$$

Remark 3.9. It is remark that the Theorem 3.8 is trivial if $x_0\in\mathscr{C}$, because the statement in the proof that $\mathscr{C}$ and the line segment $kx_0+(1-k)x$ are disjoint is no longer necessarily true if $x_0\in\mathscr{C}$.

Theorem 3.10. *Let* $\mathscr{X}$ *be a normed space and* $\mathscr{T}$ *and* $\mathscr{I}$ *self-mappings of* $\mathscr{X}$. *Let* $\mathscr{C}$ *be subset of* $\mathscr{X}$ *such that* $\mathscr{T}(\partial\mathscr{C}\cap\mathscr{C})\subset\mathscr{C}$ *and* $x_0\in F(\mathscr{T})\cap F(\mathscr{I})$. *Suppose that* $\mathscr{P}_{\mathscr{C}}(x_0)$ *is nonempty,* $\mathscr{D}$ *has a contractive family* $\mathscr{F}=\{f_\alpha\}_{\alpha\in\mathscr{D}}$, $\mathscr{I}$ *satisfies the* $property$ (Γ) *on* $\mathscr{D}$ *and* $\mathscr{I}(\mathscr{D})=\mathscr{D}$. *Suppose further that* $\mathscr{T}$ *and* $\mathscr{I}$ *are commuting on* $\mathscr{D}$ *and satisfy* (3.4) *for all* $x,y\in\mathscr{D}\cup\{x_0\}$. *If* $\mathscr{T}$ *is continuous, then* $\mathscr{P}_{\mathscr{C}}(x_0)\cap F(\mathscr{T})\cap F(\mathscr{I})\neq\phi$ *under each of the conditions of Theorem* 3.8.

Proof. Let $x\in\mathscr{D}$, then proceeding as in the proof of Theorem 3.8, we obtain $\mathscr{T}x\in\mathscr{P}_{\mathscr{C}}(x_0)$. Moreover, since $\mathscr{T}$ commutes with $\mathscr{I}$ on $\mathscr{D}$ and $\mathscr{T}$ and $\mathscr{I}$ satisfies (3.4),

$$\|\mathscr{I}\mathscr{T}x-x_0\|=\|\mathscr{T}\mathscr{I}x-\mathscr{T}x_0\|\leq\|\mathscr{I}^2x-\mathscr{I}x_0\|=\|\mathscr{I}x-x_0\|=dist(x_0,\mathscr{C}).$$

Thus $\mathcal{I}\mathcal{T}x \in \mathcal{P}_{\mathcal{C}}(x_0)$ and so $\mathcal{T}x \in \mathcal{D}_{\mathcal{C}}^{\mathcal{I}}(x_0)$. Hence $\mathcal{T}x \in \mathcal{D}$. Consequently, $\mathcal{T}(\mathcal{D}) \subset \mathcal{D} = \mathcal{I}(\mathcal{D})$. Now Theorem 3.1 and 3.2 guarantees that $\mathcal{P}_{\mathcal{C}}(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

Theorem 3.11. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of $\mathcal{X}$. Let $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Suppose that $\mathcal{P}_{\mathcal{C}}(x_0)$ is nonempty, $\mathcal{D}$ has a contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{D}}$ and $\mathcal{I}(\mathcal{C}) \cap \mathcal{D} \subset \mathcal{I}(\mathcal{D}) \subset \mathcal{D}$. Suppose further that $\mathcal{T}$ and $\mathcal{I}$ are commuting on $\mathcal{D}$ and satisfy* (3.4) *for all $x \in \mathcal{D} \cup \{x_0\}$. If $\mathcal{T}$ is continuous, then $\mathcal{P}_{\mathcal{C}}(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem* 3.8.

Proof. Let $x \in \mathcal{D}$. As in Theorem 3.8, we obtain $\mathcal{T}x \in \mathcal{D}$, that is, $\mathcal{T}(\mathcal{D}) \subset \mathcal{D}$ and $x \in \partial\mathcal{C} \cap \mathcal{C}$ and so $\mathcal{T}(\mathcal{D}) \subset \mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{I}(\mathcal{C})$. Thus, we can choose $y \in \mathcal{C}$ such that $\mathcal{T}x = \mathcal{I}y$. Because $\mathcal{I}y = \mathcal{T}x \in \mathcal{P}_{\mathcal{C}}(x_0)$, it follows that $y \in \mathcal{D}_{\mathcal{C}}^{\mathcal{I}}(x_0)$. Consequently, $\mathcal{T}(\mathcal{D}) \subset \mathcal{I}(\mathcal{D}_{\mathcal{C}}^{\mathcal{I}}(x_0)) \subset \mathcal{P}_{\mathcal{C}}(x_0)$. Therefore, $\mathcal{T}(\mathcal{D}) \subset \mathcal{I}(\mathcal{C}) \cap \mathcal{D} \subset \mathcal{I}(\mathcal{D}) \subset \mathcal{D}$. Now Theorem 3.1 and 3.2 guarantees that $\mathcal{P}_{\mathcal{C}}(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

Remark 3.12. We observe that $\mathcal{I}(\mathcal{P}_{\mathcal{C}}(x_0)) \subset \mathcal{P}_{\mathcal{C}}(x_0)$ implies $\mathcal{P}_{\mathcal{C}}(x_0) \subset \mathcal{D}_{\mathcal{C}}^{\mathcal{I}}(x_0)$ and hence $\mathcal{D} = \mathcal{P}_{\mathcal{C}}(x_0)$. Consequently, Theorem 3.8, 3.10 and 3.11 remain valid when $\mathcal{D} = \mathcal{P}_{\mathcal{C}}(x_0)$. Hence one gets the following results.

Corollary 3.13. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of $\mathcal{X}$. Let $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Suppose that $\mathcal{D} = \mathcal{P}_{\mathcal{C}}(x_0)$ is nonempty and has a contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{D}}$ and $\mathcal{I}(\mathcal{D}) = \mathcal{D}$. Suppose further that $\mathcal{I}$ satisfies the $property$ (Γ), $\mathcal{I}$ is nonexpansive on $\mathcal{D}$, $\mathcal{T}$ and $\mathcal{I}$ satisfy (3.1) for all $x \in \mathcal{D}$, some $\mathcal{R} > 0$, and $\mathcal{T}$ and $\mathcal{I}$ satisfy* (3.4)*, for all $x, y \in \mathcal{D} \cup \{x_0\}$, all $k \in (0,1)$. If $\mathcal{T}$ is continuous, then $\mathcal{P}_{\mathcal{C}}(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem* 3.8.

Corollary 3.14. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of $\mathcal{X}$. Let $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Suppose that $\mathcal{D} = \mathcal{P}_{\mathcal{C}}(x_0)$ is nonempty and has a contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{D}}$ and $\mathcal{I}(\mathcal{D}) = \mathcal{D}$. Suppose $\mathcal{I}$ is nonexpansive on $\mathcal{D}$, $\mathcal{T}$ and $\mathcal{I}$ commuting for $x \in \mathcal{D}$, and $\mathcal{T}$ and $\mathcal{I}$ satisfy* (3.4)*, for all $x \in \mathcal{D} \cup \{x_0\}$, all $k \in (0,1)$. If $\mathcal{T}$ is continuous, then $\mathcal{D} \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem* 3.8.

Corollary 3.15. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of $\mathcal{X}$. Let $\mathcal{C}$ be subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Suppose that $\mathcal{D} = \mathcal{P}_{\mathcal{C}}(x_0)$ is nonempty and has a contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{D}}$. Suppose $\mathcal{I}$ satisfies the property* (Γ)*, $\mathcal{T}$ and $\mathcal{I}$ satisfy* (3.1) *for all $x \in \mathcal{D}$, some $\mathcal{R} > 0$, and $\mathcal{T}$ is $\mathcal{I}$-nonexpansive, for all $x \in \mathcal{D} \cup \{x_0\}$, all $k \in (0,1)$, $\mathcal{I}$ is nonexpansive on $\mathcal{D}$, then $\mathcal{D} \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem* 3.8.

Remark 3.16. Remark 3.5, Theorem 3.8 − Cor 3.15 generalize Theorem 2.2 and 3.2 of Al-Thagafi [1], Theorem 3 of Sahab, Khan and Sessa [11] and Singh [14, 15] in the sense that the domain of mappings need not be starshaped, map $\mathcal{I}$ is not necessarily linear and generalized nonexpansive non-commutative maps have been used in place of relatively nonexpansive commutative maps.

Remark 3.17. Remark 3.5, Theorem 3.8 − Cor 3.15 generalize and improve related results of Shahzad [12, 13] in the sense that the domain of mappings need not be starshaped, maps $\mathscr{I}$ is not necessarily affine and mappings are not $\mathscr{R}$-subweakly commuting.

Remark 3.18. Theorem 3.8 − Cor 3.15 generalize Theorem 2 of Mukherjee and Som [10] in the sense that the two mappings are used in place of one and generalized nonexpansive non-commutative mappings have been used in place of nonexpansive of a mapping.

Remark 3.19. Theorem 3.1 − Cor 3.6 generalize the results of of Dotson [4, 5] and Habiniak [6].

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