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ON BEST APPROXIMATIONS IN THE LEBESGUE-BOCHNER SPACE $L^1(X)$

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Abstract. If $\Sigma_0 \subseteq \Sigma$ is a sub-σ-field and X a reflexive Banach space, we show that the Lebesgue-Bochner space $L^1(\Sigma_0, X)$ is proximinal in $L^1(\Sigma, X)$. Then we examine how the set of best approximations and the distance function depend on Σ_0.

1. Introduction

If V is a Banach space and W is a closed subset of it, we say that W is "proximinal" in V, if for every $v \in V$, there exists $w \in W$ such that

$$d(v, W) = \inf\{\|v - w'\| : w' \in W\} = \|v - w\|.$$

Obviously any compact subset of V is proximinal.

Let (Ω, Σ, μ) be a probability space and $\Sigma_0 \subseteq \Sigma$ a sub-σ-field of Σ. Consider the closed subspace $L^1(\Omega, \Sigma_0)$ of $L^1(\Omega, \Sigma)$. Shintani-Ando [3], proved that $L^1(\Omega, \Sigma_0)$ is proximinal in $L^1(\Omega, \Sigma)$. The purpose of this note is to extend this result of Shintani-Ando [3], to Lebesgue-Bochner spaces and establish a continuous dependence result as the sub-σ-field varies in a certain given sense.

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2. Main results

Let (Ω, Σ, μ) be a probability measure, $\Sigma_0 \subseteq \Sigma$ a sub-σ-field of Σ, X a reflexive Banach space and consider the following Lebesgue-Bochner spaces:

$$L^1(\Omega, \Sigma_0; X) = \left\{ f : \Omega \to X,\ \Sigma_0 - \text{measurable and} \int_\Omega \|f(\omega)\| d\mu(\omega) < \infty \right\}$$

and

$$L^1(\Omega, \Sigma; X) = \left\{ f : \Omega \to X,\ \Sigma - \text{measurable and} \int_\Omega \|f(\omega)\| d\mu(\omega) < \infty \right\}.$$

Here $f : \Omega \to X$ Σ-measurable means that there exists a sequence of Σ-measurable simple functions $\{s_n\}_{n\geq 1}$ s.t. $\|f(\omega) - s_n(\omega)\| \to 0$ μ-a.e. as $n \to \infty$. In the literature, the term strong measurability is often used to describe measurability.

It is clear that $L^1(\Omega, \Sigma_0, X)$ is a closed subspace of $L^1(\Omega, \Sigma; X)$. The next theorem extends the result of Shintani-Ando [3].

Theorem 1. *$L^1(\Omega, \Sigma_0; X)$ is proximinal in $L^1(\Omega, \Sigma; X)$.*

Proof. Let $f \in L^1(\Omega, \Sigma; X)$. Then for $g \in L^1(\Omega, \Sigma_0; X)$ and $A \in \Sigma_0$, we have:

$$\int_A \|g(\omega)\| d\mu(\omega) \leq \int_A \|f(\omega) - g(\omega)\| d\mu(\omega) + \int_A \|f(\omega)\| d\mu(\omega)$$

If $\chi_{A^c}(\cdot)$ is the characteristic function of $A^c \in \Sigma_0$ (i.e. $\chi_{A^c}(\omega) = 1$ if $\omega \in A^c$ and $\chi_{A^c}(\omega) = 0$ if $\omega \in A$), then we have

$$\int_\Omega \|f(\omega) - \chi_{A^c}(\omega) g(\omega)\| d\mu(\omega) = \int_{A^c} \|f(\omega) - g(\omega)\| d\mu(\omega) + \int_A \|f(\omega)\| d\mu(\omega).$$

Set $d(f, \Sigma_0) = \inf\{\int_\Omega \|f(\omega) - g'(\omega)\| d\mu(\omega) : g' \in L^1(\Omega, \Sigma_0; X)\}$. Then since $\chi_{A^c}(\cdot) g(\cdot) \in L^1(\Omega, \Sigma_0; X)$, we have that

$$\int_A \|g(\omega)\| d\mu(\omega) + d(f, \Sigma_0)$$
$$\leq \int_A \|g(\omega)\| d\mu(\omega) + \int_\Omega \|f(\omega) - \chi_{A^c}(\omega) g(\omega)\| d\mu(\omega)$$

$$\le \int_A \|f(\omega) - g(\omega)\| d\mu(\omega) + \int_A \|f(\omega)\| d\mu(\omega) + \int_\Omega \|f(\omega) - \chi_{A^c}(\omega) g(\omega)\| d\mu(\omega)$$
$$\le \int_A \|f(\omega) - g(\omega)\| d\mu(\omega) + \int_{A^c} \|f(\omega) - g(\omega)\| d\mu(\omega) + 2\int_A \|f(\omega)\| d\mu(\omega). \quad (1)$$

Now let $\{g_n\}_{n\ge 1} \subseteq L^1(\Omega, \Sigma_0; X)$ be a minimizing sequence for our best approximation problem; i.e. $\|f - g_n\|_{L^1(X)} \downarrow d(f, \Sigma_0)$ as $n \to \infty$. From inequality (1) above, we know that for every $A \in \Sigma_0$ and every $n \ge 1$, we have

$$\int_A \|g_n(\omega)\| d\mu(\omega) + d(f, \Sigma_0) \le \int_\Omega \|f(\omega) - g_n(\omega)\| d\mu(\omega) + 2\int_A \|f(\omega)\| d\mu(\omega).$$

So if $\{A_n\}_{n\ge 1} \subseteq \Sigma_0$ is such that $\mu(A_n) \downarrow 0$, then we have

$$\overline{\lim} \int_{A_n} \|g_n(\omega)\| d\mu(\omega) + d(f, \Sigma_0)$$
$$\le \lim \int_\Omega \|f(\omega) - g_n(\omega)\| d\mu(\omega) + 2 \lim \int_{A_n} \|f(\omega)\| d\mu(\omega).$$

But recall that $\{g_n\}_{n\ge 1} \subseteq L^1(\Omega, \Sigma_0; X)$ is a minimizing sequence. So $\int_\Omega \|f(\omega) - g_n(\omega)\| \, d\mu(\omega) = \|f - g_n\|_{L^1(\Sigma, X)} \to d(f, \Sigma_0)$ as $n \to \infty$, while clearly $\lim \int_{A_n} \|f(\omega)\| d\mu(\omega) = 0$. Therefore

$$\lim \int_{A_n} \|g_n(\omega)\| g\mu(\omega) \;=\; 0.$$

Also it is clear that $\{g_n\}_{n\ge 1}$ is bounded in $L^1(\Omega, \Sigma_0; X)$. Therefore $\{g_n\}_{n\ge 1}$ is uniformly integrable in $L^1(\Omega, \Sigma_0; X)$. Finally note that because of the reflexivity of X, for each $A \in \Sigma_0$, $\{\int_A g_n(\omega) d\mu(\omega)\}$ is bounded, hence relatively weakly compact. So we can apply theorem 1, p. 101 of Diestel-Uhl [1] and deduce that $\{g_n\}_{n\ge 1}$ is relatively weakly compact in $L^1(\Omega, \Sigma_0; X)$ and by the Eberlein-Smulian theorem relatively sequentially weakly compact in $L^1(\Omega, \Sigma_0; X)$. So by passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g$ in $L^1(\Omega, \Sigma_0; X)$. Then $g \in L^1(\Omega, \Sigma_0; X)$ and since the norm is weakly lower semicontinuous, we have

$$\|f - g\|_{L^1(\Sigma, X)} \le \underline{\lim} \|f - g_n\|_{L^1(\Sigma, X)} = d(f, \Sigma_0)$$

so that $\|f-g\|_{L^1(\Sigma,X)} = d(f,\Sigma_0)$.

Hence $L^1(\Omega,\Sigma_0;X)$ is proximinal in $L^1(\Omega,\Sigma,X)$. ∎

In what follows, by $E^{\Sigma}nf$ (resp. $E^{\Sigma_0}f$) we will denote the conditional expectation of $f \in L^1(\Omega,\Sigma;X)$ with respect to Σ_n (resp. Σ_0). Let $\{\Sigma_n\}_{n\geq 1} \subseteq \Sigma$ be a sequence of sub-σ-fields of Σ. We say that $\Sigma_n \overset{L^1(\Sigma,X)}{\longrightarrow} \Sigma_0$ if and only if for every $f \in L^1(\Omega,\Sigma;X)$, $E^{\Sigma_n}f \overset{s}{\longrightarrow} E^{\Sigma_0}f$ in $L^1(\Omega,\Sigma;X)$. Clearly if $\Sigma_n \uparrow \Sigma_0$, then from the vector-valued extension of Levy's theorem (see Metivier [2], theorem 11.2), we have that $\Sigma_n \overset{L^1(\Sigma,X)}{\longrightarrow} \Sigma_0$.

Let $f \in L^1(\Omega,\Sigma;X)$ and set

$$P_n(f) = \{g \in L^1(\Omega,\Sigma_n;X) : d(f,\Sigma_n) = \|f-g\|_{L^1(\Sigma,X)}\}, \quad n \geq 1$$

$$\text{and } P(f) = \{g \in L^1(\Omega,\Sigma_0;X) : d(f,\Sigma_0) = \|f-g\|_{L^1(\Sigma,X)}\}.$$

These are nonempty by Theorem 1.

Define $w-\overline{\lim}P_n(f) = \{g \in L^1(\Omega,\Sigma;X) : g = w\text{-}\lim g_{n_k},\ g_{n_k} \in P_{n_k}(f),\ n_1 < n_2 < \ldots < n_k < \ldots\}$. Then we have the following convergence (continuous dependence) result.

Theorem 2. *If $\Sigma_n \uparrow \Sigma_0$, then $w\text{-}\overline{\lim}P_n(f) \subseteq P(f)$ in $L^1(\Omega,\Sigma;X)$ and $d(f,\Sigma_n) \to d(f,\Sigma_0)$ as $n \to \infty$.*

Proof. Let $g \in w\text{-}\overline{\lim}P_n(f)$. Then by definition and by denoting subsequences with the same index as sequences, we know that we can find $g_n \in P_n(f)$ s.t. $g_n \overset{w}{\longrightarrow} g$ in $L^1(\Omega,\Sigma;X)$. Since $g \in L^1(\Omega,\Sigma_n;X)$ and $\Sigma_n \uparrow \Sigma_0$, we see that $g \in L^1(\Omega,\Sigma_0;X)$. Let $h \in L^1(\Omega,\Sigma_0;X)$. Then $E^{\Sigma_n}h \in L^1(\Omega,\Sigma_n;X)$ and from theorem 11.2, p. 16 of Metivier [2], we have $E^{\Sigma_n}h \overset{s}{\longrightarrow} E^{\Sigma_0}h = h$ in $L^1(\Omega,\Sigma;X)$. Then exploiting the weak lower semicontinuity of the norm, we have

$$\|f-g_n\|_{L^1(\Sigma,X)} = d(f,\Sigma_n) \leq \|f-E^{\Sigma_n}h\|_{L^1(\Sigma,X)}$$

$$\text{and } \|f-g\|_{L^1(\Sigma,X)} \leq \underline{\lim}\|f-g_n\|_{L^1(\Sigma,X)}$$

$$\leq \lim\|f-E^{\Sigma_n}h\|_{L^1(\Sigma,X)} = \|f-h\|_{L^1(\Sigma,X)}.$$

Since $h \in L^1(\Omega, \Sigma_0; X)$ was arbitrary, we deduce that

$$\|f - g\|_{L^1(\Sigma,X)} \leq d(f, \Sigma_0).$$

But $g \in L^1(\Omega, \Sigma_0; X)$. So

$$\|f - g\|_{L^1(\Sigma,X)} = d(f, \Sigma_0),$$

that is, $g \in P(f)$.

Therefore $w\text{-}\overline{\lim} P_n(f) \subseteq P(f)$in $L^1(\Omega, \Sigma; X)$.

Also note that we have shown that

$$d(f, \Sigma_0) \;=\; \|f - g\|_{L^1(\Sigma,X)} \leq \underline{\lim} \|f - g_n\|_{L^1(\Sigma,X)} \;=\; \underline{\lim} d(f, \Sigma_n) \qquad (2)$$

On the other hand, for any $h \in L^1(\Omega, \Sigma_0; X)$

$$d(f, \Sigma_n) \;=\; \|f - g_n\|_{L^1(\Sigma,X)} \leq \|f - E^{\Sigma_n} h\|_{L^1(\Sigma,X)}$$

and hence $\overline{\lim} d(f, \Sigma_n) \leq \|f - h\|_{L^1(\Sigma,X)}$.

It follows that $\overline{\lim} d(f, \Sigma_n) \leq d(f, \Sigma_0)$. (3)

From (2) and (3) above, we conclude that

$$d(f, \Sigma_n) \to d(f, \Sigma_0) \text{ as } n \to \infty.$$

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