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SOME REMARKS ON MULTIPLICATON MODULES*

CHANG WOO CHOI AND EUN SUP KIM

1. Introduction

Throughout this thesis all rings are commutative rings with an identity and all modules are unital.

Let R be a ring and M an R-module. Then M is called a *multiplication module* provided for each submodule N of M there exists an ideal I of R such that $N = IM$.

If N is a submodule of M then $(N : M) = \{r \in R : rM \subseteq N\}$. It is clear that every cyclic R-module is a multiplication module. In particular, invertible, and more generally projective, ideals of R are multiplication R-modules (see [9, Theorem 1]).

Let M be an R-module. If P is a maximal ideal of R, then we define

$$T_P(M) \;=\; \{m \in M : (1-p)m = 0 \text{ for some } p \in P\}.$$

Clearly $T_P(M)$ is a submodule of M. We say that M is P-cyclic provided there exist $q \in P$ and $m \in M$ such that $(1-q)M \subseteq Rm$. On the other hand M is called *P-torsion provided for each $m \in M$* there exists $p \in P$ such that $(1-p)m = 0$.

In this paper we investigate the relationships between multiplication module and its dual module. In particular, we prove the Fitting's Lemma in terms of multiplication module. Also, we define $\theta(M) = \sum_{m \in M}(Rm : M)$ for any R-module M and show that some properties of multiplication module by the technique $\theta(M)$.

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2. Multiplication modules

Let R be a commutative ring with identity and M an R-module. Then M is called a *multiplication module* if for each submodule N of M there exists an ideal I of R such that $N = IM$.

It is easy to check that M is a multiplication module if and only if $N = (N : M)M$ for every submodule N of M. An R-module M is called a *locally cyclic* if M_P is a cyclic R_P-module for all maximal ideals P of R. Our starting point is the following result taken from [4, Theorem 1.2].

Lemma 2.1. *An R-module M is a multiplication module if and only if for every maximal ideal P of R either $M = T_P(M)$ or M is P-cyclic.*

Proposition 2.2. *Let K and L be submodules of an R-module M and let N be a multiplication submodule of M which is not P-cyclic for every maximal ideal P of R. Then $L \cap (K + N) \subseteq (L \cap K) + PL$.*

Proof. Let $a \in L \cap (K + N)$. Then there exist $b \in K$ and $c \in N$ such that $a = b + c$. Since N is a multiplication submodule of M which is not P-cyclic, $N = T_P(N)$ by Lemma 2.1. Therefore for all $n \in N$, there exists $p \in P$ such that $(1 - p)n = 0$ and hence $(1 - p)c = 0$. Thus $(1 - p)a = (1 - p)(b + c) = (1 - p)b + (1 - p)c = (1 - p)b$ and so $a = (1 - p)a + pa = (1 - p)b + pa$. But $(1 - p)b = (1 - p)a \in L \cap K$. This implies $a = (1 - p)b + pa \in (L \cap K) + PL$. Hence $L \cap (K + N) \subseteq (L \cap K) + PL$.

Proposition 2.3. *Let M be a multiplication R-module and P a maximal ideal of R. Then the following statements are equivalent.*

(i) *M is P-torsion.*

(ii) *$M = PM$.*

(iii) *$ann(m) + P = R$ for all $m \in M$.*

Proof. (i) $\Longleftrightarrow$ (ii). It is well-known ([8, Lemma 2.4]).

(i) $\Rightarrow$ (iii). Suppose M is P-torsion. Then there exists $p \in P$ such that

$(1-p)m = 0$ for all $m \in M$. This implies $1-p \in ann(m)$ and so $1 \in ann(m)+P$. Hence $ann(m) + P = R$ for all $m \in M$.

(iii) $\Rightarrow$ (i). Suppose $ann(m) + P = R$ for all $m \in M$. Then $1 = a + q$ for some $a \in ann(m)$ and $q \in P$. This means $m = qm$ and so $(1 - q)m = 0$. Thus M is P-torsion.

Remark. In the proof of the equivalence between (i) and (iii), the condition that M is a multiplication module is not necessary.

Corollary 2.4. *An R-module M is a multiplication module if and only if for each maximal ideal P of R either $ann(m) + P = R$ for all $m \in M$ or M is P-cyclic.*

Proof. By Proposition 2.3 and Lemma 2.1, it is obvious.

Let R be a ring and M an R-module. The module M will be called a *torsion module* in case $ann(m) \neq 0$ for every $m \in M$, otherwise it will be called *non-torsion.* Note that any non-torsion module is faithful. For the ring R let $C(R)$ denote the set of regular elements (i.e. non-zero-divisors,), so $c \in C(R)$ if any only if $cr \neq 0$ for all $0 \neq r \in R$. Let F denote the ring of quotients of R, so that every element of F can be written in the form $c^{-1}r$ for some $c \in C(R)$, $r \in R$. If I is an ideal of R let $I^+ = \{f \in F : fI \subseteq R\}$. The ideal I is called *invertible* provided $I^+I = R$. It is well known and easy to prove that an ideal I of R is invertible if and only if it is a multiplication ideal which contains a regular element of R.

An R-module M is called *torsion-free* provided $cm \neq 0$ for all $c \in C(R)$, $0 \neq m \in M$. The dual M^* of M is defined to be $\mathrm{Hom}_R(M, R)$ which is itself an R-module with respect to the definitions: $(\alpha + \beta)(m) = \alpha(m) + \beta(m)$ and $(r\alpha)(m) = r\alpha(m)$, for all $\alpha, \beta \in M^*$, $r \in R$ and $m \in M$.

Theorem 2.5. *Let R be a domain and let M be a nonzero multiplication R-module. Then M is torsion free if and only if M^* is a non-torsion module.*

Proof. Let $0 \neq y \in M$. There exists an ideal I of R such that $Ry = IM$.

Clearly $I \neq 0$. Let $0 \neq x \in I$. Define $\phi : M \to Ry$ by $\phi(m) = xm (m \in M)$. Since M is torsion free, ϕ is a monomorphism. But $Ry \cong R$. Thus M is isomorphic to an ideal A of R. This implies that A is a multiplication ideal containing a regular element of R and hence an invertible ideal of R. Let a be the regular element of R contained in A. Then $ann(a) = 0$ and hence $ann(A) = 0$. Thus A is a faithful R-module and so M is a faithful R-module because $M \cong A$. By [6, Lemma 1.1], M^* is a non-torsion module.

Conversely, suppose M^* is a non-torsion module. By definition, M^* is a faithful R-module. But $ann(M) \subseteq ann(M^*)$. For, let $y \in ann(M)$ and $\phi \in M^*$. Then $y \in R$, $yM = 0$ and so $\phi(yM) = \phi(0) = 0$. This implies $y\phi(M) = (y\phi)M = 0$ and so $y\phi = 0$ and hence $yM^* = 0$ i.e. $y \in ann(M^*)$. Since $ann(M^*) = 0$, $ann(M) = 0$ i.e. M is faithful. By [4, Lemma 4.1], M is torsion free.

Corollary 2.6. *Let R be a domain and let M be a non zero multiplication R-module. If M is torsion free, then M^* is a finitely generated faithful multiplication R-module and $M^{**} \cong M$.*

Proof. By the proof of Theorem 2.5 and [6, Theorem 1.3 Corollary 2], it follows immediately.

Corollary 2.7. *Let R be a domain and let M be a non zero multiplication R-module. If M is torsion free, then M is a finitely generated non-torsion module.*

Proof. By the proof of Theorem 2.5 and [6, Theorem 1.7], it is clear.

Definition 2.8. A module M is said to satisfy *Fitting's Lemma* if for each $f \in End_R(M)$ there exists an integer $n \geq 1$ such that $M = \mathrm{Ker} f^n \oplus \mathrm{Im} f^n$.

Definition 2.9. A ring R is *left (right) π-regular* if for each a in R there exists b in R and an integer $n \geq 1$ such that $a^n = ba^{n+1} (a^n = a^{n+1}b)$.

Theorem 2.10. *Let M be a multiplication R-module satisfying descending chain condition on multiplication submodules and $f \in End_R(M)$. Then M*

satisfies Fitting's Lemma.

Proof. Consider the sequence $M \supset f(M) \supset f^2(M) \cdots$. Since every homomorphic images of multiplication modules are multiplication module, the sequence becomes stationary after n steps, say. Thus $f^{(n)}(M) = f^{(n+1)}(M) = \cdots = f^{2n}(M) = \cdots$. Therefore f^n induces an endomorphism on multiplication module $f^{(n)}(M)$ which is an epimorphism, hence an automorphism because every epimorphism of a multiplication module onto itself is an automorphism. Thus $f^{(n)}(M) \cap \mathrm{Ker} f^{(n)} = 0$. Now take any $m \in M$, then $f^n(m) = f^{2n}(n)$ for some $n \in M$, hence $m - f^{(n)}(n) \in \mathrm{Ker}(f^n)$. But $m = f^n(n) + (m - f^n(n))$. Thus $M = f^{(n)}(M) \oplus \mathrm{Ker} f^n$. This completes the proof.

Following Azumaya [3], a ring which is both left and right π-regular will be called strongly π-regular. However, F. Dischinger proved that every left π-regular rings are right π-regular (and hence stronly π-regular).

Corollary 2.11. *If a multiplication R-module M satisfies descending chain condition on multiplication submodules and* $f \in End_R(M)$, *then* $End_R(M)$ *is strongly* π*-regular.*

Proof. By Theorem 2.10 and [2, Proposition 2.3].

Corollary 2.12. *If a multiplication R-module M satisfies the hypothesis of corollary* 2.11, *then every injective or surjective endomorphisms of M are isomorphisms.*

Proof. By Corollary 2.11 and [2, Corollary 2.4].

Let R be a ring and M an R-module. A submodule X of M is said to be *fully invariant* if $f(X) \subseteq X$, for all $f \in \mathrm{Hom}_R(M, M)$.

Lemma 2.13. *Let M be a multiplication R-module. Then every submodule of M is fully invariant.*

Proof. Let N be any submodule of M. Then $N = IM$ for some ideal I of R. Let $f \in \mathrm{End}_R(M, M)$. Then $f(N) = f(IM) = If(M) \subseteq IM = N$.

A. G. Naoum [7] has shown that if M is a finitely generated multiplication module then for each $f \in \mathrm{End}_R(M, M)$, there exists $r \in R$ such that $f(m) = rm$ for each $m \in M$. Our next theorem generalizing this fact communicated to me in a personal correspondence by Patrick F. Smith.

Theorem 2.14. *Let M be a multiplication R-module and let $f \in \mathrm{End}_R(M)$. Then there exists $r \in R$ such that $f(m) = rm$ for each $m \in M$.*

Proof. It follows from Lemma 2.13.

Corollary 2.15. *If M is a multiplication R-module, then $\mathrm{End}_R(M)$ is isomorphic to $R/ann(M)$.*

Proof. Define $\psi : R \to \mathrm{End}_R(M)$ by $\psi(r) = f_r$ for any $r \in R$, where $f_r(m) = rm$ for any $m \in M$. It can easily be checked that ψ is a ring homomorphism and by Theorem 2.14, ψ is onto.

A little elementary calculation shows that $\mathrm{Ker}(\psi) = \mathrm{ann}(M)$. Thus End_R (M) is isomorphic to $R/\mathrm{ann}(M)$.

3. Multiplication module and $\theta(M)$

For given an R-module M, we consider the associated ideal $\theta(M) = \sum_{m \in M}$ $(Rm : M)$ and we shall be concerned with relationships between the ideal $\theta(M)$ of a commutative ring R and multiplication modules.

Lemma 3.1 [5]. *A finitely generated module is a multiplication module if and only if it is locally cyclic.*

Theorem 3.2. *Let M be a finitely generated R-module. Then M is a multiplication module if and only if $\theta(My) = R$ for all $y \in R$.*

Proof. Suppose M is a multiplication module. Then M is locally cyclic by Lemma 3.1 and hence My is locally cyclic for all $y \in R$. For, let P be any maximal ideal of R and let $y \in R$. Then,

$$\begin{aligned}(My)_P &= (MRy)_P = M_P(Ry)_P \\ &= M_P R_p y \\ &= \left(\frac{a}{s}\right) y \text{ for some } \frac{a}{s} \in M_P \text{ because } M \text{ is locally cyclic.} \\ &= \left(\frac{ay}{s}\right).\end{aligned}$$

Since $\frac{ay}{s} \in (My)_P$, My is locally cyclic for all $y \in R$. Clearly My is finitely generated. Hence $\theta(My) = R$ by [1, Theorem 1].

Conversely, suppose $\theta(My) = R$ for all $y \in R$. In particular, $\theta(M) = R$ and hence M is finitely generated and locally cyclic by [1, Theorem 1]. By Lemma 3.1, M is a multiplication module.

Corollary 3.3. *Let M be a finitely generated R-module. Then M is a multiplication module if and only if My is a multiplication module for all $y \in R$.*

Proof. By the proof of Theorem 3.2, My is finitely generated and locally cyclic. Hence My is a multiplication module by Lemma 3.1.

Proposition 3.4. *Let M be a finitely generated multiplication R-module. Then there exists a finitely generated ideal I contained in $\theta(M)$ such that $M = IM$.*

Proof. Since M is a multiplication R-module, $M = \theta(M)M$. For $x \in M$, $Rx = (Rx : M)M$. Hence $M = \sum_{x \in M} Rx = \sum_{x \in M}(Rx : M)M = \theta(M)M$. Say $n_1, \ldots, n_n$ be the generators of M. Then

$$\begin{aligned}n_1 &= a_1 m_1 + \ldots + a_n m_n \\ &\vdots \\ n_n &= b_1 m_1' + \ldots + b_n m_n'\end{aligned}$$

where $a_i, b_i \in \theta(M)$ and $m_i, m_i' \in M$ for all $1 \leq i \leq n$. Take $I = (a_1, \ldots, a_n) + \cdots + (b_1, \ldots, b_n)$. Then I is a finitely generated ideal contained in $\theta(M)$ such that $M = IM$. This completes the proof.

Proposition 3.5. *Let M be a nonzero multiplication R-module such that $M \neq IM$, for every proper ideal I of R. Then $\theta(M) = R$.*

Proof. Suppose $M \neq IM$ for every proper ideal I of R. Let $x \in M$. Then Rx is a submodule of M. Since M is a multiplication R-module, $Rx = (Rx : M)M$. Thus $M = \sum_{x \in M} Rx = \sum_{x \in M}(Rx : M)M = (\sum_{x \in M}(Rx : M))M$ and hence $\sum_{x \in M}(Rx : M) = R$ by hypothesis. Therefore $\theta(M) = R$.

Corollary 3.6. *Let M be a nonzero multiplication R-module such that $M \neq IM$ for every proper ideal I of R. Then M is finitely generated and locally cyclic.*

Proof. By Proposition 3.5 and [1, Theorem 1], it is obvious.

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Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea.