TAMKANG JOURNAL OF MATHEMATICS
Volume 24, Number 3, Autumn 1993

ON RINGS SATISFYING BOTH OF 1-*abc* AND 1-*cba* BEING INVERTIBLE OR NONE

CHEN-TE YEN

Abstract. Let R be a ring with identity 1 and n a positive integer. We define the property P_n as follows: (P_n) If $1 - a_1a_2a_3 \dots a_{n-1}a_n$ is invertible in R, then so is $1 - a_na_2a_3 \dots a_{n-1}a_1$. Thus, R satisfies (P_n), for some $n \geq 3$ if and only if R satisfies (P_3). Some properties of rings satisfying (P_3) are obtained, e.g., R must be directly finite.

Throughout the paper R will denote an associative ring with identity 1, and Perlis-Jacobson radical $J(R)$. We define the property P_n as follows:

(P_n) If $1 - a_1a_2a_3 \dots a_{n-1}a_n$ is invertible in R, then so is $1 - a_na_2a_3 \dots a_{n-1}a_1$.

An element a in a ring R is invertible if there exists an element b of R such that $ab = ba = 1$; and idempotent if $a^2 = a$. R is regular (strongly regular) if for each a in R there exists an b of R such that $aba = a$ ($a^2b = a$). R is semisimple if $J(R) = 0$. R is called prime if $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$. R is called directly finite ([1], p. 166) if $ab = 1$ in R implies $ba = 1$.

It is easy to see that R always satisfies (P_2):

Lemma 1. ([3], p. 89, Exercise 4). *If $1 - ab$ is invertible in R, then so is $1 - ba$.*

Proof. The inverse of $1 - ba$ is $1 + b(1 - ab)^{-1}a$.

Received May, 5, 1992; revised July 15, 1992.
1980 Mathematics Subject Classification (1985 Revision). Primary 16A99.
Key words and phrases: Regular, semisimple, prime ring, Perlis-Jacobson radical, directly finite.

Theorem 1. *Let R satisfy (P_n). If $1 - a_1 a_2 a_3 \ldots a_{n-1} a_n$ is invertible in R, then so are $1 - a_{f(1)} a_{f(2)} a_{f(3)} \ldots a_{f(n-1)} a_{f(n)}$ for all f in S_n, the symmetric group of degree n.*

Proof. We know that S_n is generated by the cycles $(12, \cdots, n)$ and $(1n)$. Since R satisfies (P_n), by Lemma 1 the result follows.

Since R has 1, R satisfies (P_n) for some $n \geq 3$ if and only if R satisfies (P_3). So, in the sequel we assume that R satisfy (P_3). The class of rings with (P_3) is closed under isomorphic images, and finite direct sums.

Trivially, any field satisfies (P_3), this is the only division rings:

Lemma 2. *If R is a division ring and satisfies (P_3), then R is a field.*

Proof. For all $a \neq 0$, $b \neq 0$ in R, we have $1 - (a^{-1}b^{-1})ba = 0$. Thus by (P_3), $1 - ab(a^{-1}b^{-1}) = 0$. Hence, $ab = ba$.

Lemma 3. *If R is a simple Artinian ring and satisfies (P_3), then R is a field.*

Proof. By Wedderburn-Artin Theorem ([2], Theorem 2.1.6), $R \cong D_n$, the ring of all $n \times n$ matrices over a division ring D. Thus, D_n satisfies (P_3). Suppose $n > 1$. Then $1 - (e_{11} + e_{21})e_{12}e_{21} = 1 - e_{11} - e_{21}$ is not invertible, but $1 - e_{21}e_{12}(e_{11} + e_{21}) = 1 - e_{21}$ is invertible, a contradiction. Hence, $n = 1$ and so $R \cong D$ a division ring. By Lemma 2, R is a field.

Theorem 2. *If R is a semisimple Artinian ring and satisfies (P_3), then R is isomorphic to a direct sum of a finite number of fields.*

Proof. By ([2], Theorem 2.1.7), $R \cong D_{n_1}^{(1)} \oplus \cdots \oplus D_{n_k}^{(k)}$, where the $D^{(i)}$ are division rings and where $D_{n_i}^{(i)}$ is the ring of all $n_i \times n_i$ matrices over $D^{(i)}$. It is easy to see that each $D_{n_i}^{(i)}$ satisfies (P_3). So by Lemma 3, each $D_{n_i}^{(i)}$ is a field.

Lemma 4. *Let A be an ideal of $R, A \subseteq J(R)$ and $\overline{R} = R/A$. If $a \in R$, then $\overline{a} = a + A$ is invertible in $\overline{R} \iff a$ is invertible in R.*

Proof. $(\Leftarrow)$: Trivial.

$(\Rightarrow)$: Let $\overline{a} \in \overline{R}$ and $\overline{a}$ be invertible in $\overline{R}$. Then there exists an $\overline{b}$ in $\overline{R}$ such that $\overline{a}\,\overline{b} = \overline{1} = \overline{b}\,\overline{a}$. Thus $1 - ab \in A$ and $1 - ba \in A$. Since$A \subseteq J(R)$, by ([5], p. 57, Proposition 5), both of ab and ba are invertible in R. Hence, a has a left and right inverse and so a is invertible.

Theorem 3. *Let A be an ideal of R, $A \subseteq J(R)$ and $\overline{R} = R/A$. Then R satisfies $(P_3) \Longleftrightarrow \overline{R}$ satisfies (P_3).*

Proof. Let $a, b, c \in R$.

$(\Rightarrow)$: If $\overline{1} - \overline{a}\,\overline{b}\,\overline{c}$ is invertible in $\overline{R}$, then $1 - abc$ is invertible in R by Lemma 4. Thus, $1 - cba$ is invertible. So, by Lemma 4 again $\overline{1} - \overline{c}\,\overline{b}\,\overline{a}$ is invertible.

$(\Leftarrow)$: The implication of $(\Rightarrow)$ is reversed.

Noncommutative rings satisfying (P_3) actually exist, e. g., Examples 1 and 2 of ([6]) or the following:

Example. Let $\mathbb{Z}_3$ be the field of integers modulo 3.

Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_3 \right\}.$$

Then R is a noncommutative ring with identity. Since $R/J(R) \cong \mathbb{Z}_3$, R satisfies (P_3) by Theorem 3.

Combining Theorems 2 and 3 yields

Theorem 4. *If R is Artinian and satisfies (P_3), then $R/J(R)$ is isomorphic to a direct sum of a finite number of fields.*

Lemma 5. *If R satisfies (P_3), then R is directly finite.*

Proof. Let $a, b \in R$ and $ab = 1$. Then $1 - a(1 - ba)b = 1$ implies $1 - ab(1 - ba) = ba$ is invertible by Theorem 1. Thus, there exists an c in R such that $(ba)c = 1$. Hence, $ac = (ab)ac = a(ba)c = a$ and so $ba = 1$.

Lemma 6. *If R is a strongly regular ring, then R is regular. (This is a well-known result.)*

Proof. It is easy to see that strong regularity implies that R has no nonzero nilpotent elements. Let $a \in R$. Then there exists an b in R such that $a^2b = a$. Let $e = ab$. Thus, $ae = a$ and so $(ea - a)^2 = 0$. Hence, $ea - a = 0$ and so $aba = a$.

Lemma 7. *If R is a semisimple ring and satisfies (P_3), then each idempotent in R is central; and for $a, b, c \in R$, $aba = a = aca$ implies $ab = ba$, $aR(ba - 1) = 0$ and $aR(b - c) = 0$.*

Proof. Let $e^2 = e \in R$. Then for all x, y in R, $1 - e(ex - x)y = 1$ implies $1 - (ex - x)ey$ is invertible by Theorem 1. By ([5], p. 57, Proposition 3), $(ex - x)e \in J(R) = 0$ and so $exe = xe$. Similarly, we can show that $exe = ex$. Hence, $ex = xe$ for all x in R. So, e is central.

Let $a, b, c \in R$ and $aba = a = aca$. Then $(ba)^2 = b(aba) = ba$, and so by the result above we have $aR(ba - 1) = 0$. Let $f = ba$, $g = ac$ and $h = ab$. Then for all x in R, $1 - a(1 - ba)xb = 1$ implies $1 - (1 - ba)abx$ is invertible by Theorem 1 again. Thus, we get $(1 - f)h = (1 - ba)ab \in J(R) = 0$ and so $h = fh$. Similarly, we can show that $f = fh$. Therefore, $ab = h = fh = f = ba$. Since $(ac)^2 = ac$, by the results above we have $hg = (ab)ac = (aba)c = ac = g$ and $hg = (ab)ac = (ac)ab = (aca)b = ab = h$. Thus, $ab = ac$. Then for all x, y in R, $1 - a(b - c)xy = 1$ implies $1 - ax(b - c)y$ is invertible by Theorem 1. Hence, we get $ax(b - c) \in J(R)$ and so $ax(b - c) = 0$. Therefore, $aR(b - c) = 0$.

Theorem 5. *Let R satisfy (P_3). Then R is regular $\Longleftrightarrow$ R is strongly regular.*

Proof. ($\Rightarrow$): Let R be regular. Then by ([4], p. 111, Theorem 21), R is semisimple. Using Lemma 7, we conclude that R is strongly regular.

($\Leftarrow$): The implication is Lemma 6.

We end this paper with

Theorem 6. *If R is a prime and regular ring and satisfies (P_3), then R is a field.*

Proof. By ([4], p. 111, Theorem 21), R is semisimple. Let $0 \neq a, b \in R$ and $aba = a$. Then by Lemma 7, $aR(ba-1) = 0$. This yields $ba - 1 = 0$ by primeness of R. Applying Lemma 5, we have $ab = 1$. So, R is a division ring. By Lemma 2, R is a field.

For the related results, see [7] and [8].

References

[1] K. R. Goodearl, "Ring theory", *Marcel Dekker Inc*, New York and Basel, 1976.

[2] I. N. Herstein, "Noncommutative rings", *Carus Math. Monographs*, No. 15, Math. Assoc. of Amer., 1971.

[3] N. Jacobson, "Basic algebra 1", *San Francisco*, W. H. Freeman, 1974.

[4] I. Kaplansky, "Fields and rings", *Univ. of Chicago Press*, Chicago, 1972.

[5] J. Lambek, "Lecture on rings and modules", *Waltham*, Mass.: Blaisdell, 1966.

[6] C. T. Yen, "On the commutativity of primary rings", *Math. Japonica*, 25 (1980), 449-452.

[7] C. T. Yen, "A condition for simple ring implying field", *Acta Mathematica Hungarica*, 61(1993), 51-52.

[8] C. T. Yen, "Another condition on simple ring implying field", unpublished manuscript.

Department of Mathematics, Chung Yuan University, Chung Li, Taiwan, 320, Republic of China.