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A NOTE ON SPACES VIA DENSE SETS

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Abstract. Some spaces have been defined depending on the concept of dense set in a given topological space (X, τ) such as: resolvable space, irresolvable space, hereditarily irresolvable space, and submaximal space. We study many of their properties and explore several relationships between these spaces and SMPC function which has been defined recently as a dual of the concept of precontinuity.

1. Introduction

In this paper, we will use the following notational conventions: a "space" will always mean a topological space; given a space (X, τ), for and $A \subseteq X$, we denote by Int$A$ and ClA the interior of A and the closure of A with respect to τ respectively, and "iff" for "if and only if".

A space (X, τ) is resolvable [1] if there is a dense subset $D \subseteq X$ for which $X - D$ is also dense. A space which is not resolvable is called irresolvable. A subset of X is resolvable (irresolvable) if it is resolvable (irresolvable) as a subspace. A space is hereditarily irresolvable if each of its nonempty subsets is irresolvable. Such spaces were investigated by Hewitt [2] where it was shown

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(see also Theorem 1 of [1]) that every space (X,τ) can be expressed as a disjoint union $F \cup G$ with F closed and resolvable and G hereditarily irresolvable. The F and G are unique and $F \cup G$ is called the Hewitt representation of X. A space (X,τ) is said to be submaximal if each of its dense subsets are open. Clearly every submaximal space is irresolvable and in fact hereditarily irresolvable. A is called preopen [4] if $A \subseteq \mathrm{Int}ClA$, and $PO(X,\tau)$ means the collection of all preopen sets in (X,τ). For any space (X,τ) let τ_P be the smallest topology on X containing $PO(X,\tau)$. The topology τ^α [5] is $PO(X,\tau) \cap SO(X,\tau)$ where $A \in SO(X,\tau)$ iff A is semi-open [6]. i.e. $A \subseteq Cl\mathrm{Int}A$. Thus, for any space (X,τ), $\tau \subseteq \tau^\alpha \subseteq PO(X,\tau) \subseteq \tau_P$. It is also known that $PO(X,\tau^\alpha) = PO(X,\tau)$ (Corollary 1 of [7]). A function $f : (X,\tau) \to (Y,\sigma)$ is said to be precontinuous [4], preirresolute [8] and strongly M-precontinuous [9] (SMPC) if the inverse image of each open, preopen and preopen in (Y,σ) is preopen, preopen and open in (X,τ), respectively.

2. On Resolvability

Theorem 1. *Each semi-open subset of a resolvable space is resolvable.*

Proof. Let $A \in SO(X,\tau)$, i.e. $A \subseteq Cl\mathrm{Int}A \subseteq X$ and X is resolvable then $\mathrm{Int}A$ is resolvable and A-$\mathrm{Int}A$ is nowhere dense in $(A,\tau/A)$. Thus, if $D_1 \cup D_2$ is a disjoint union of dense subsets of $\mathrm{Int}A$ then $D_0 = D_1 \cup (A - \mathrm{Int}A)$ and D_2 are disjoint and also are dense in A.

Lemma 1. (*Corollary* 5 *of* [1]) *If (X,τ) is resolvable then $\tau_P = 2^X$.*

Proof. Let D_1 and D_2 be disjoint dense subsets of X and let $x \in X$. Then $D_1 \cup \{x\}$ and $D_2 \cup \{x\}$ are dense and hence preopen. Thus $\{x\} = (D_1 \cup \{x\}) \cap D_2 \cup \{x\}) \in \tau_P$.

Lemma 2. *$f : (X,\tau) \to (Y,\sigma)$ is SMPC iff $f : (X,\tau) \to (Y,\sigma_P)$ is continuous.*

Proof. A basic open set in σ_P has the form $V = \bigcap_{k=1}^{n} B_k$ where each $B_k \in PO(Y,\sigma)$. So if $f:(X,\tau) \to (Y,\sigma)$ is SMPC, and V is a basic open set in σ_P, $f^{-1}(V) = \bigcap_{k=1}^{n} f^{-1}(B_k) \in \tau$ so that $f:(X,\tau) \to (Y,\sigma_P)$ is continuous. The converse is clear since $PO(Y,\sigma) \subseteq \sigma_P$.

Theorem 2. *If either* (1) *every open subset of* Y *is closed, or* (2) (Y,σ) *is resolvable then*

$f:(X,\tau) \to (Y,\sigma)$ *is SMPC iff* $f:(X,\tau) \to (Y,2^Y)$ *is continuous.*

Proof. By Lemmata 1 and 2 and the foregoing remarks, in either case, $\sigma_P = 2^Y$.

We offer the following consequences.

Corollary 1. *If* (Y,σ) *is resolvable, the following are equivalent.*

i. $f:(X,\tau) \to (Y,\sigma)$ *is SMPC.*
ii. $f^{-1}(B)$ *is clopen* (*closed and open*) *for each* $B \subseteq Y$.
iii. $f^{-1}(y)$ *is clopen for each* $y \in Y$.
iv. $f^{-1}(y)$ *is open for each* $y \in Y$.
v. $f:(X,\tau) \to (Y,2^Y)$ *is continuous.*

Corollary 2. *If* (X,τ) *is connected and* (Y,σ) *is resolvable then* $f:(X,\tau) \to (Y,\sigma)$ *is SMPC iff* f *is a constant function.*

For example if R is the usual space of real numbers, every nonconstant function $f: R \to R$ is not SMPC.

Corollary 3. *If* (X,τ) *is dense-in-itself* (*has no isolated points*) *and* (Y,σ) *is a nonempty resolvable space then there is no SMPC injection* $f:(X,\tau) \to (Y,\sigma)$.

Theorem 3. *If* (X,τ) *is a space,* $X = X_1 \cup X_2$ *and* $X_1 \cap X_2 = \phi$ *and* X_1 *is closed then if both* $(X_1,\tau/X_1)$ *and* $(X_2,\tau/X_2)$ *are hereditarily irresolvable then* (X,τ) *is hereditarily irresolvable.*

Proof. Suppose that $\phi \neq A \subseteq X$ and $(A, \tau/A)$ is resolvable. Then there exist disjoint, dense in A, subsets D_1 and D_2 with $A = D_1 \cup D_2$. Suppose that $D_1 \cap X_2 \neq \phi$, and $D_2 \cap X_2 \neq \phi$. Then since X_2 is open in X, $D_1 \cap X_2$ and $D_2 \cap X_2$ are disjoint and dense in $A \cap X_2$. For if $x \in D_2 \cap X_2$ and V is open with $x \in V$, since D_1 is dense in A, $V \cap A \cap D_1 \neq \phi$. If U is open in X and $x \in U$ then, for $V = U \cap X_2$, $V \in \tau$ and $x \in V$ so that $U \cap A \cap (D_1 \cap X_2) \neq \phi$. Thus, $D_1 \cap X_2$ and similarly $D_2 \cap X_2$ are dense in $A \cap X_2$ and disjoint. Thus, $A \cap X_2$ is a resolvable subspace of X_2 which contradicts X_2 being hereditarily irresolvable. Apparently, either $D_1 \cap X_2 = \phi$ or $D_2 \cap X_2 = \phi$. But in either case $A \cap X_1$ contains a dense set in A. Thus, $Cl_A(A \cap X_1) = A \subseteq A \cap X_1 \subseteq X_1$ since X_1 is closed. Thus A is a resolvable subspace of X_1 which cannot be since X_1 is hereditarily irresolvable. This final contradiction proves that (X, τ) is hereditarily irresolvable.

We also note that every subspace of a hereditarily irresolvable space is hereditarily irresolvable.

3. On Submaximality

Proposition 1. *For a submaximal space (X, τ), if ρ is a finer topology than τ on X. Then (X, ρ) is also submaximal.*

Proof. If $D \subseteq X$ is ρ-dense then $X = Cl_\rho D \subseteq Cl_\tau D$ this leads to D is τ-dense and hence $D \in \tau$. Thus, $D \in \rho$ showing that (X, ρ) is submaximal.

Theorem 4. *Submaximality is preserved by open surjections.*

Proof. If $f : (X, \tau) \to (Y, \sigma)$ is an open surjection and (X, τ) is submaximal and if $D \subseteq Y$ is dense, $f^{-1}(D)$ is dense and hence open in X so that $D = f(f^{-1}(D))$ is open.

Corollary 4. *If ΠX_α is submaximal then each X_α is submaximal.*

Now, we show that open and hence semi-open subsets of a submaximal space are submaximal. We first note the following useful known lemma.

Lemma 3. *If $A \in SO(X,\tau)$ then $\tau^\alpha/A = (\tau/A)^\alpha$.*

Theorem 5. *If (X,τ) is submaximal and $A \in SO(X,\tau)$ then $(A,\tau/A)$ is submaximal.*

Proof. Since (X,τ) is submaximal and A $SO(X,\tau)$. Then $\tau = \tau^\alpha$ and there is an open, dense, hereditarily irresolvable subset $D \subseteq X$. If $A \neq \phi$, then $D \cap \mathrm{Int}A$ is a dense, open, hereditarily irresolvable subspace of $(A,\tau/A)$, and also $Cl_A(D \cap \mathrm{Int}A) = A \cap Cl(D \cap \mathrm{Int}A) = A \cap Cl\,\mathrm{Int}A = A$. Since $\tau/A = \tau^\alpha/A = (\tau/A)^\alpha$, we have that $(A,\tau/A)$ is submaximal.

Lemma 4. (Proposition 1 of [1]) *$A \in PO(X,\tau)$ iff $A = U \cap D$ for some $U \in \tau$ and dense $D \subseteq X$.*

Proof. $A \in PO(X,\tau) \to A \subseteq \mathrm{Int}\,ClA = U \in \tau$. Let $D = X - (U - A) = (X - U) \cup A$. Then D is dense since $X = ClA \cup (X - ClA) \subseteq ClA \cup (X - U) = ClD$. Also, $A = U \cap D$. Conversely, if $A = U \cap D$ with $U \in \tau$ and D dense, $A \subseteq U \subseteq \mathrm{Int}\,ClU = \mathrm{Int}\,Cl(A)$ so that $A \in PO(X,\tau)$.

Lemma 5. *If (X,τ) is submaximal then $PO(X,\tau) = \tau$.*

Proof. Clearly $\tau \subseteq PO(X,\tau)$. Now $A \in PO(X,\tau) \to A = U \cap D$ for some $U \in \tau$ and dense $D \subseteq X$. Therefore, if (X,τ) is submaximal, $D \in \tau \to A \in \tau$. Clearly the three parts of the next theorem follow from lemma 5.

Theorem 6. *For $f : (X,\tau) \to (Y,\sigma)$, the following holds*

(i) *If (X,τ) is submaximal, then f is SMPC iff it is preirresolute.*

(ii) *SMPC coincides with continuity if (Y,σ) is submaximal.*

(iii) *If both (X,τ) and (Y,σ) are submaximal, then SMPC, preirresolute precontinuity and continuity are equivalent.*

4. On SMPC

Lemma 6. (Theorem 5 of [1]) *For a space (X,τ) let $X = F \cup G$ denote the*

Hewitt-representation of (X,τ). *Then* $PO(X,\tau)$ *is a topology on* X *iff* $Cl\,G$ *is open and* $\{x\} \in PO(X,\tau)$ *for each* $x \in \operatorname{Int} F$.

Theorem 7. *If a space* (Y,σ) *as* (X,τ) *in Lemma 6, then* $f : (X,\tau) \to (Y,\sigma)$ *is SMPC iff* $f : (X,\tau) \to (Y, PO(Y,\sigma))$ *is continuous.*

It was shown in Proposition 3.4 of [10] and independently in Theorem 1 of [7] that every precontinuous semi-open function is preirresolute, where a function is semi-open if images of open sets are semi-open. Consequently projections on product spaces are always preirresolute being both continuous and open. These suggests simpler proofs for next results in which we will abbreviate a space (X,τ) by X and $\{X_\alpha : \alpha \in \nabla\}$ means the family of topological spaces.

Proposition 2. *If* $f : X \to \Pi X_\alpha$ *is SMPC, then* $p_\alpha f : X \to X_\alpha$ *is SMPC, for each* $\alpha \in \nabla$ *(where* p_α *is the projection of* ΠX_α *onto* X_α*, for each* $\alpha \in \nabla$*).*

Proof. Since $f : X \to \Pi X_\alpha$ is SMPC, then each p_α is preirresolute, each $p_\alpha \circ f$ is SMPC by Theorem 3.3 (v) (1) [9].

Corollary 5. *Let* $f_\alpha : X \to X_\alpha$*,* $\alpha \in \nabla$ *be a class of functions defined as* $f_\alpha(x) = x_\alpha$ *and* $f : X \to \Pi X_\alpha$ *is given by* $f(x) = \{f_\alpha(x)\}$ *for each* $x \in X$ *and* $\alpha \in \nabla$. *If* f *is SMPC then* f_α *is SMPC, for each* $\alpha \in \nabla$.

Proof. By previous proposition, and the fact that each $f_\alpha = p_\alpha \circ f$.

Theorem 8. *Each function of the family* $f_\alpha : X_\alpha \to Y_\alpha$*,* $\alpha \in \nabla$ *is SMPC if the function* $f : \Pi X_\alpha \to \Pi Y_\alpha$*, which is defined by* $f\{x_\alpha\} = \{f_\alpha(x_\alpha)\}$ *is SMPC.*

Proof. Since f is SMPC, then each $q_\alpha \circ f : \Pi X_\alpha \to Y_\alpha$ is SMPC by Proposition 2 where $q_\alpha : \Pi Y_\alpha \to Y_\alpha$ is the projection. Then if $p_\alpha : \Pi X_\alpha \to X_\alpha$, since $f_\alpha \circ p_\alpha = q_\alpha$, $f_\alpha \circ p_\alpha$ is SMPC. Now since p_α is open, by Theorem 3.3 (ii) [9], f_α is SMPC.

The converse of Theorem 8 may not be hold in general, as the following example illustrates.

Example 1. Let $X = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \cdots\}$ have the usual real subspace topology. Then the only proper dense subset of X is $D = \{\frac{1}{n} : n = 1, 2, \cdots\}$ which is open so that X is submaximal. By Lemma 5 above or Theorem 3.1 [9], the identity function $1_X : X \to X$ is SMPC. However, $1_X \times 1_X = 1_X^2 : X^2 \to X^2$ is the identity function on the product space X^2 and is not SMPC. For $\{(0,0)\} \cup (D \times D)$ is dense and hence preopen in X^2 but not open. Consequently, also, X^2 is not submaximal. However, X^2 is hereditarily irresolvable for if $Y_1 = \{0\} \times X \cup X \times \{0\}$ and $Y_2 = X^2 - Y_1$, then it is easily seen that as subspaces of X^2, Y_1 and Y_2 are each hereditarily irresolvable and further Y_1 is closed in X^2. The result follows from Theorem 3.

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