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INEQUALITIES OF HARDY TYPE IN TWO VARIABLES

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1. Introduction

In 1920, Hardy [1] (or [3], Theorem 327) proved the following theorem:

Theorem A. *If* $p > 1$, $f(x) \geq 0$ *for* $0 < x < \infty$ *and* $G(x) = \frac{1}{x}\int_0^x f(t)dt$, *then*

$$\int_0^\infty G^p(x)dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,$$

unless $f \equiv 0$. *The constant is best possible.*

In 1928, Hardy [2] proved the following generalization of (1.1):

Theorem B. *If* $p > 1$, $m \neq 1$, $f(x) \geq 0$ *for* $0 < x < \infty$, *and*

$$G(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x f(t)dt, & \text{if } m > 1, \\ \dfrac{1}{x}\displaystyle\int_x^\infty f(t)dt, & m < 1, \end{cases}$$

then

$$\int_0^\infty \chi^{p-m} G^p(x)dx < \left(\frac{p}{m-1}\right)^p \int_0^\infty x^{-m}\big(xf(x)\big)^p dx, \quad \text{if} \quad m > 1, \tag{1.2}$$

and

$$\int_0^\infty \chi^{p-m} G^p(x)dx < \left(\frac{p}{1-m}\right)^p \int_0^\infty x^{-m}\big(xf(x)\big)^p dx, \quad \text{if} \quad m < 1, \tag{1.2'}$$

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unless $f \equiv 0$, *the constant is best possible.*

Here functions are assumed to be measurable and left sides of inequalities exist when right sides do. In 1963, Levinson [4] proved the following generalization of (1.1):

Theoem C. *For* $x > 0$, *let* $f(x) \geq 0$ *and* $r(x) > 0$ *be absolutely continuous. If* $p > 1$ *and* $\lambda > 0$ *so that* $\frac{p-1}{p} + \frac{xr'(x)}{r(x)} \geq \frac{1}{\lambda}$, *for almost all* $x > 0$, *then*

$$\int_0^\infty G^p(x)dx \leq \lambda^p \int_0^\infty f^p(x)dx, \tag{1.3}$$

where $G(x) = \frac{1}{xr(x)} \int_0^x r(t)f(t)dt$. *The case* $r = 1$ *and* $\lambda = \frac{p}{p-1}$, *shows the constant to be best possible. In* 1986, *K. L. Lee and G. S. Yang established the following generalizations:*

Theorem D. *Let* $p > 1$ *and* f, r *be defined as in Theorem* C. *If there is a* $\lambda > 0$, *such that*

$$p - 1 + \frac{R(x)r'(x)}{r^2(x)} - \frac{(p-m)R(x)}{xr(x)} \geq \frac{p}{\lambda},$$

for almost all $x > 0$, *then*

$$\int_0^\infty x^{-m+p} \left(\frac{1}{R(x)} \int_0^x r(t)f(t)dt\right)^p dx \leq \lambda^p \int_0^\infty x^{-m+p} f^p(x)dx, \quad m > 1, \tag{1.4}$$

where $R(x) = \int_0^x r(t)dt$.

Theorem E. *Let* $p > 1$ *and* f, r *be defined as in Theorem* C. *If there is a* $\lambda > 0$ *such that*

$$\frac{m-1}{p} + \frac{xr'(x)}{r(x)} \geq \frac{1}{\lambda},$$

for almost all $x > 0$, *then*

$$\int_0^\infty x^{-m+p} G^p(x)dx \leq \lambda^p \int_0^\infty x^{-m+p} f^p(x)dx, \quad m > 1, \tag{1.5}$$

where $G(x) = \frac{1}{xr(x)} \int_0^x r(t)f(t)dt$.

Theorem F. *Let $p > 1$, $\alpha > 1$ and f, r be defined as in Theorem C. If there is a $\lambda > 0$ such that*

$$p - 1 - \frac{(m + \alpha p - p)R(x)}{xr(x)} - \frac{R(x)r'(x)}{r^2(x)} \geq \frac{p}{\lambda},$$

for almost all $x > 0$, then

$$\int_0^\infty x^{-m-\alpha p+p} \left(\frac{1}{R(x)} \int_x^\infty t^\alpha r(t) f(t) dt\right)^p dx \leq \lambda^p \int_0^\infty x^{-m+p} f^p(x) dx, \ m < 1, \tag{1.6}$$

where

$$R(x) = \int_x^\infty r(t)dt.$$

Theorem G. *Let $p > 1$, $\alpha > 1$ and f, r be defined as in Theorem C. If there is a $\lambda > 0$ such that*

$$\frac{1-m}{p} - \frac{xr'(x)}{r(x)} - \alpha \geq \frac{1}{\lambda},$$

for almost all $x > 0$, then

$$\int_0^\infty x^{-m} G^p(x) dx \leq \lambda^p \int_0^\infty x^{-m+p} f^p(x) dx, \quad m < 1, \tag{1.7}$$

where $G(x) = \frac{1}{x^\alpha r(x)} \int_x^\infty t^\alpha r(t) f(t) dt$.

In this paper, we shall establish inequalities similar to (1.4) – (1.7) in case the function f considered is a function of two variables.

2. The case $m > 1$

Theorem 2.1. *Let $p > 1$, f be a nonnegetive integrable function defined on $\{(x, y) : x > 0, y > 0\}$ and q, r be positive absolutely continuous on $(0, \infty)$. If there are $\alpha > 0$, $\beta > 0$ such that*

$$p - 1 + \frac{Q(x)q'(x)}{q^2(x)} - \frac{(p-m)Q(x)}{xq(x)} \geq \frac{p}{\alpha}, \quad \textit{for all } x \in (0, \infty) \tag{2.1}$$

$$p - 1 + \frac{R(y)r'(y)}{r^2(y)} - \frac{(p-m)R(y)}{yr(y)} \geq \frac{p}{\beta}, \quad \textit{for all } y \in (0, \infty) \tag{2.2}$$

where $Q(x) = \int_0^x q(s)ds$ *and* $R(y) = \int_0^y r(t)dt$, *then*

$$\int_0^\infty \int_0^\infty (xy)^{-m+p} \left(\frac{1}{Q(x)R(y)} \int_0^x \int_0^y q(s)r(t)f(s,t)ds\,dt \right)^p dxdy$$
$$\leq (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y)dx\,dy. \tag{2.3}$$

Proof. If $0 < a < b$, $0 < c < d$, let

$$I(x,y) = \int_a^x \int_c^y q(s)r(t)f(s,t)ds\,dt \; x \in [a,b], \; y \in [c,d].$$

Then

$$\int_a^b \int_c^d (xy)^{-m+p} \left(\frac{1}{Q(x)R(y)} \int_a^x \int_c^y q(s)r(t)f(s,t)ds\;dt \right)^p dx\,dy$$
$$= \int_a^b \left[\int_c^d (xy)^{-m+p} (Q(x)R(y))^{-p} I^p(x,y)dx \right] dy$$
$$= \int_c^d y^{-m+p}R^{-p}(y) \left[\int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y)dx \right] dy. \tag{2.4}$$

Fix y and integration by parts, we have

$$\int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y)dx = \frac{Q^{-p+1}(x)x^{-m+p}}{(-p+1)q(x)} I^p(x,y)\Big|_a^b$$
$$+ \frac{1}{p-1} \int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y) \left[-\frac{Q(x)q'(x)}{q^2(x)} + \frac{(p-m)Q(x)}{xq(x)} \right] dx$$
$$+ \frac{p}{p-1} \int_a^b x^{-m+p}Q^{-p+1}(x)I^{p-1}(x,y) \left(\int_c^y r(t)f(x,t)dt \right) dx.$$

so that

$$\int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y) \left[p - 1 + \frac{Q(x)q'(x)}{q^2(x)} - \frac{(p-m)Q(x)}{xq(x)} \right] dx$$
$$\leq p \int_a^b x^{-m+p}Q^{-p+1}(x)I^{p-1}(x,y) \left(\int_c^y r(t)f(x,t)dt \right) dx.$$

Using (2.1) and Holder inequality with indices p and $p/(p-1)$, we have

$$\int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y)dx \tag{2.5}$$

$$\le \alpha \int_a^b \left(x^{-m+p}Q^{-p}(x)I^p(x,y)\right)^{\frac{p-1}{p}} \left[x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p dx\right]^{\frac{1}{p}}$$

$$\le \alpha \left[\int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y)dx\right]^{\frac{p-1}{p}} \left[\int_a^b x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p dx\right]^{\frac{1}{p}}.$$

It follows that $\int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y)dx \le \alpha^p \int_a^b x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p dx$. Substituting this inequality in (2.4) and using Fubini's theorem, we have

$$\int_c^d y^{-m+p}R^{-p}(y)\left[\int_a^b x^{-m+p}Q^{-p}(x)I^p(x,y)dx\right]dy$$

$$\le \alpha^p \int_c^d y^{-m+p}R^{-p}(y)\left[\int_a^b x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p dx\right]dy$$

$$= \alpha^p \int_a^b x^{-m+p}\left[\int_c^d y^{-m+p}R^{-p}(y)\left(\int_c^y r(t)f(x,t)dt\right)^p dy\right]dx. \tag{2.6}$$

Fix x and apply Theorem D, we have

$$\int_c^d y^{-m+p}R^{-p}\left(\int_c^y r(t)f(x,t)dt\right)^p dy \le \beta^p \int_c^d y^{-m+p}f^p(x,y)dy.$$

Using this inequality in (2.6), we obtain

$$\int_a^b \int_c^d (xy)^{-m+p}\left(Q(x)R(y)\right)^{-p} I^p(x,y)dx\,dy$$

$$\le (\alpha,\beta)^p \int_a^b \int_c^d (xy)^{-m+p}f^p(x,y)dx\,dy$$

$$\le (\alpha,\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p}f^p(x,y)dx\,dy.$$

Let $a < s < b$ and $c < t < d$. Then

$$\int_s^b \int_t^d (xy)^{-m+p}\left(Q(x)R(y)\right)^{-p} I^p(x,y)dx\,dy$$

$$\le (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p}f^p(x,y)dx\,dy.$$

Let $a \to 0$ and $c \to 0$, we see that

$$\int_s^b \int_t^d (xy)^{-m+p} \left(\frac{I}{Q(x)R(y)} \int_0^x \int_0^y q(s)r(t)f(s,t)ds\,dt \right)^p dx\,dy$$
$$\le (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y)dx\,dy.$$

holds for all s, t, b, d such that $0 < s < b$ and $0 < t < d$, the required inequality then follows.

Theorem 2.2. *Let $p > 1$ and f, g, r be defined as in Theorem 2.1. If there are $\alpha > 0$, $\beta > 0$ such that*

$$\frac{m-1}{p} + \frac{xq'(x)}{q(x)} \ge \frac{1}{\alpha}, \quad \text{for all} \quad x \in (0,\infty), \tag{2.7}$$

$$\frac{m-1}{p} + \frac{yr'(y)}{r(y)} \ge \frac{1}{\beta}, \quad \text{for all} \quad y \in (0,\infty), \tag{2.8}$$

then

$$\int_0^\infty \int_0^\infty (xy)^{-m+p} \left(\frac{1}{xyq(x)r(y)} \int_0^x \int_0^y q(s)r(t)f(s,t)dsdt \right)^p dx\,dy$$
$$\le (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y)dx\,dy. \tag{2.9}$$

Proof. If $0 > a > b$, $0 < c < d$, let

$$I(x,y) = \int_0^x \int_0^y q(s)r(t)f(s,t)ds\,dt, x \in [a,b], \ y \in [c,d]$$

then

$$\int_a^b \int_c^d (xy)^{-m+p} \left(\frac{1}{xyq(x)r(y)} \int_a^x \int_c^y q(s)r(t)f(s,t)ds\,dt \right)^p dx\,dy$$
$$= \int_a^b \left[\int_c^d (xy)^{-m+p} \left(\frac{1}{xyq(x)r(y)} \right)^p I^p(x,y)dx \right] dy$$
$$= \int_c^d y^{-m} r^{-p}(y) \left[\int_c^b x^{-m} q^{-p}(x) I^p(x,y)dx \right] dy. \tag{2.10}$$

Fix y, integration by parts gives

$$\int_a^b x^{-m} q^{-p}(x) I^p(x,y)dx = \frac{x^{-m+1}}{-m+1} q^{-p}(x) I^p(x,y)\Big|_a^b$$
$$+ \frac{p}{m-1}\int_a^b x^{-m} q^{-p}(x) I^p(x,y)\left(\frac{-xq'(x)}{q(x)}\right)dx$$
$$+ \frac{p}{m-1}\int_a^b x^{-m+1} q^{-p+1}(x) I^{p-1}(x,y)\left(\int_c^y r(t)f(x,t)dt\right)dx.$$

so that

$$\int_a^b x^{-m} q^{-p}(x) I^p(x,y)\left[\frac{m-1}{p} + \frac{xq'(x)}{q(x)}\right]dx$$
$$\leq \int_a^b x^{-m+1} q^{-p+1}(x) I^{p-1}(x,y)\left(\int_c^y r(t)f(x,t)dt\right)dx.$$

Applying (2.7) and Holder inequality with indices p and $p/(p-1)$, we have

$$\int_a^b x^{-m} q^{-p}(x) I^p(x,y)dx \tag{2.11}$$
$$\leq \alpha\int_a^b \left(x^{-m} q^{-p}(x) I^p(x,y)\right)^{\frac{p-1}{p}}\left[x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p\right]^{\frac{1}{p}}dx$$
$$\leq \alpha\left[\int_a^b x^{-m} q^{-p}(x) I^p(x,y)dx\right]^{\frac{p-1}{p}}\left[\int_a^b x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p dx\right]^{\frac{1}{p}}.$$

It follows that

$$\int_a^b x^{-m} q^{-p}(x) I^p(x,y)dx \leq \alpha^p\int_a^b x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p dx.$$

Substituting this inequality in (2.10) and using Fubini's theorem, we have

$$\int_c^d y^{-m} r^{-p}(y)\left[\int_a^b x^{-m} q^{-p}(x) I^p(x,y)dx\right]dy$$
$$\leq \alpha^p\int_c^d y^{-m} r^{-p}(y)\left[\int_a^b x^{-m+p}\left(\int_c^y r(t)f(x,t)dt\right)^p dx\right]dy$$
$$= \alpha^p\int_a^b x^{-m+p}\left[\int_c^d y^{-m} r^{-p}(y)\left(\int_c^y r(t)f(x,t)dt\right)^p dy\right]dx. \tag{2.12}$$

Now fix x and use Theorem E, we have

$$\int_c^d y^{-m} r^{-p}(y) \left(\int_c^y r(t) f(x,t) dt \right)^p dy \leq \beta^p \int_c^d y^{-m+p} f^p(x,y) dy.$$

Substituting this inequality in (2.12), we have

$$\begin{aligned} &\int_a^b \int_c^d (xy)^{-m} \left(\frac{1}{q(x)r(y)} \right)^p I^p(x,y) dx\, dy \\ &\leq (\alpha\beta)^p \int_a^b \int_c^d (xy)^{-m+p} f^p(x,y) dx\, dy \\ &\leq (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y) dx\, dy. \end{aligned}$$

Let $a < s < b$ and $c < t < d$. Then

$$\begin{aligned} &\int_s^b \int_t^d (xy)^{-m} \left(\frac{1}{q(x)r(y)} \right)^p I^p(x,y) dx\, dy \\ &\leq (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y) dx\, dy. \end{aligned}$$

Let $a \to 0$ and $c \to 0$. We see that

$$\begin{aligned} &\int_s^b \int_t^d (xy)^{-m+p} \left(\frac{1}{xyq(x)r(y)} \int_0^x \int_0^y q(s) r(t) f(s,t) ds\, dt \right)^p dx\, dy \\ &\leq (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y) dx\, dy. \end{aligned}$$

holds for all s, b, t, d such that $0 < s < b$, $0 < t < d$. The required inequality then follows.

3. The case $m < 1$

Theorem 3.1. *Let $p > 1$, $\lambda > 1$ and f be a nonnegative integrable function on $\{(x,y) : x > 0, y > 0\}$ and q, r be positive absolutely continuous functions on $(0,\infty)$. If there are $\alpha > 0$, $\beta > 0$ such that*

$$p - 1 - \frac{(m + \lambda p - p)Q(x)}{xq(x)} - \frac{Q(x)q'(x)}{q^2(x)} \geq \frac{p}{\alpha}, \quad \textit{for all } x \in (0,\infty), \tag{3.1}$$

$$p-1-\frac{(m+\lambda p-p)R(y)}{yr(y)}-\frac{R(y)r'(y)}{r^2(y)}\geq\frac{p}{\beta},\quad \textit{for all } y\in(0,\infty), \tag{3.2}$$

where $Q(x)=\int_x^\infty q(s)ds$ *and* $R(y)=\int_y^\infty r(t)dt$, *then*

$$\int_0^\infty\int_0^\infty (xy)^{-m-\lambda p+p}\left(\frac{1}{Q(x)R(y)}\int_x^\infty\int_y^\infty (st)^\lambda q(s)r(t)f(s,t)ds\,dt\right)^p dx\,dy$$
$$\leq(\alpha\beta)^p\int_0^\infty\int_0^\infty (xy)^{-m+p}f^p(x,y)dx\,dy. \tag{3.3}$$

Proof. If $0<a<b$, $0<c<d$, let

$$I(x,y)\;=\;\int_x^b\int_y^d (st)^\lambda q(s)r(t)f(s,t)ds\,dt \qquad x\in[a,b],\quad y\in[c,d]$$

Then

$$\int_a^b\int_c^d (xy)^{-m-\lambda p+p}\left(\frac{1}{Q(x)R(y)}\int_x^b\int_y^d (st)^\lambda q(s)r(t)f(s,t)ds\,dt\right)^p dx\,dy$$
$$=\int_a^b\left[\int_c^d (xy)^{-m-\lambda p+p}\left(Q(x)R(y)\right)^{-p}I^p(x,y)dx\right]dy$$
$$=\int_c^d y^{-m-\lambda p+p}R^{-p}(y)\left[\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)dx\right]dy. \tag{3.4}$$

Fix y and integration by parts, we have

$$\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)dx\;=\;\frac{Q^{-p+1}(x)x^{-m-\lambda p+p}}{(p-1)q(x)}I^p(x,y)\Big|_a^b$$
$$-\frac{1}{p-1}\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)\left[\frac{(-m-\lambda p+p)Q(x)}{q(x)x}-\frac{Q(x)q'(x)}{q^2(x)}\right]dx$$
$$+\frac{p}{p-1}\int_a^b x^{-m-\lambda p+p}x^\lambda Q^{-p+1}(x)\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)I^{p-1}(x,y)dx.$$

so that

$$\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)\left[p-1-\frac{(m+\lambda p-p)Q(x)}{xq(x)}-\frac{Q(x)q'(x)}{q^2(x)}\right]dx$$
$$\leq p\int_a^b x^{-m-\lambda p+p}x^\lambda Q^{-p+1}(x)\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)I^{p-1}(x,y)dx.$$

Using (3.1) and Holder inequality to obtain

$$\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)dx \tag{3.5}$$

$$\leq \alpha\int_a^b \left(x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)\right)^{\frac{p-1}{p}}\left[x^{-m-\lambda p+p}x^{\lambda p}\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p\right]^{\frac{1}{p}}dx$$

$$\leq \alpha\left[\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)dx\right]^{\frac{p-1}{p}}\left[\int_a^b x^{-m+p}\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dx\right]^{\frac{1}{p}}.$$

so that

$$\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)dx \;\leq\; \alpha^p\int_a^b x^{-m+p}\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dx.$$

Substituting this inequality in (3.4) and using Fubini's theorem, we have

$$\int_c^d y^{-m-\lambda p+p}R^{-p}(y)\left[\int_a^b x^{-m-\lambda p+p}Q^{-p}(x)I^p(x,y)dx\right]dy$$

$$\leq \alpha^p\int_c^d y^{-m-\lambda p+p}R^{-p}(y)\left[\int_a^b x^{-m+p}\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dx\right]dy$$

$$= \alpha^p\int_a^b x^{-m+p}\left[\int_c^d y^{-m-\lambda p+p}R^{-p}(y)\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dy\right]dx. \tag{3.6}$$

Fix x and use Theorem F, we have

$$\int_c^d y^{-m-\lambda p+p}R^{-p}(y)\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dy \;\leq\; \beta^p\int_c^d y^{-m+p}f^p(x,y)dy.$$

Substituting this inequality in (3.6), we have

$$\int_a^b\int_c^d (xy)^{-m-\lambda p+p}\left(Q(x)R(y)\right)^{-p}I^p(x,y)dx\,dy$$

$$\leq (\alpha\beta)^p\int_a^b\int_c^d (xy)^{-m+p}f^p(x,y)dx\,dy$$

$$\leq (\alpha\beta)^p\int_0^\infty\int_0^\infty (xy)^{-m+p}f^p(x,y)dx\,dy.$$

Let $a < s < b$ and $c < t < d$. Then

$$\int_a^s \int_c^t (xy)^{-m-\lambda p+p} \left(Q(x)R(y)\right)^{-p} I^p(x,y)dx\,dy$$
$$\leq (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y)dx\,dy.$$

Let $b \to \infty$ and $d \to \infty$ we see that

$$\int_a^s \int_c^t (xy)^{-m-\lambda p+p} \left(\frac{1}{Q(x)R(y)} \int_x^\infty \int_y^\infty (st)^\lambda q(s)r(t)f(s,t)ds\,dt\right)^p dx\,dy$$
$$\leq (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y)dx\,dy,$$

holds for all a, s, c, t such that $0 < a < s$, $0 < c < t$, the required inequality then follows.

Theorem 3.2. *Let p, λ and f, r be defined as in Theorem 3.1, If there are $\alpha > 0$, $\beta > 0$, such that*

$$\frac{1-m}{p} - \frac{xq'(x)}{q(x)} - \lambda \geq \frac{1}{\alpha}, \quad \text{for all} \quad x \in (0,\infty) \tag{3.7}$$
$$\frac{1-m}{p} - \frac{yr'(y)}{r(y)} - \lambda \geq \frac{1}{\beta}, \quad \text{for all} \quad y \in (0,\infty) \tag{3.8}$$

then

$$\int_0^\infty \int_0^\infty (xy)^{-m} H^p(x,y)dx\,dy$$
$$\leq (\alpha\beta)^p \int_0^\infty \int_0^\infty (xy)^{-m+p} f^p(x,y)dx\,dy.$$

where

$$H(x,y) = \frac{1}{(xy)^\lambda q(x)r(y)} \int_x^\infty \int_y^\infty (st)^\lambda q(s)r(t)f(s,t)dsdt,$$

Proof. If $0 < a < b$, $0 < c < d$, let

$$I(x,y) = \int_x^b \int_y^d (st)^\lambda q(s)r(t)f(s,t)ds\,dt, \quad x \in [a,b],\ y \in [c,d],$$

Then

$$\int_a^b \int_c^d (xy)^{-m} \left(\frac{1}{(xy)^\lambda q(x) r(y)} \int_x^b \int_y^d (st)^\lambda q(s) r(t) f(s,t) ds dt \right)^p dx\, dy$$

$$= \int_a^b \int_c^d (xy)^{-m} \left(\frac{1}{(xy)^\lambda g(x) r(y)} \right)^p I^p(x,y) dx\, dy$$

$$= \int_c^d y^{-m} \left(\frac{1}{y^\lambda r(y)} \right)^p \left[\int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) dx \right] dy. \tag{3.10}$$

Fix y and integration by parts gives

$$\int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) dx = \frac{x^{-m+1}}{-m+1} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) \Big|_a^b$$
$$+ \frac{p}{m-1} \int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) \left(-\lambda + \frac{-x q'(x)}{q(x)} \right) dx$$
$$+ \frac{-p}{m-1} \int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^{p-1} I^{p-1}(x,y) \left(x \int_y^d t^\lambda r(t) f(x,t) dt \right) dx.$$

so that

$$\int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) \left[\frac{1-m}{p} - \frac{x q'(x)}{q(x)} - \lambda \right] dx$$
$$\leq \int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^{p-1} I^{p-1}(x,y) \left(x \int_y^d t^\lambda r(t) f(x,t) dt \right) dx.$$

Applying (3.7) and Holder inequality to obtain

$$\int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) dx \tag{3.11}$$
$$\leq \alpha \int_a^b \left[x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) \right]^{\frac{p-1}{p}} \left[x^{-m+p} \left(\int_y^d t^\lambda r(t) f(x,t) dt \right)^p \right]^{\frac{1}{p}} dx$$
$$\leq \alpha \left[\int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) dx \right]^{\frac{p-1}{p}} \left[\int_a^b x^{-m+p} \left(\int_y^d t^\lambda r(t) f(x,t) dt \right)^p dx \right]^{\frac{1}{p}}.$$

so that

$$\int_a^b x^{-m} \left(\frac{1}{x^\lambda q(x)} \right)^p I^p(x,y) dx \leq \alpha^p \int_a^b x^{-m+p} \left(\int_y^d t^\lambda r(t) f(x,t) dt \right)^p dx.$$

Substituting this inequality in (3.10) and using Fubini's theorem, we have

$$\int_c^d y^{-m}\left(\frac{1}{y^\lambda r(y)}\right)^p\left[\int_a^b x^{-m}\left(\frac{1}{x^\lambda q(x)}\right)I^p(x,y)dx\right]dy \tag{3.12}$$

$$\leq \alpha^p\int_c^d y^{-m}\left(\frac{1}{y^\lambda r(y)}\right)^p\left[\int_a^b x^{-m+p}\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dx\right]dy$$

$$= \alpha^p\int_a^b x^{-m+p}\left[\int_c^d y^{-m}\left(\frac{1}{y^\lambda r(y)}\right)^p\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dy\right]dx.$$

Fix x and use Theorem G, we have

$$\int_c^d y^{-m}\left(\frac{1}{y^\lambda r(y)}\right)^p\left(\int_y^d t^\lambda r(t)f(x,t)dt\right)^p dy \;\leq\; \beta^p\int_c^d y^{-m+p}f^p(x,y)dy.$$

Substituting this inequality in (3.15), we have

$$\int_a^b\int_c^d (xy)^{-m}\left(\frac{1}{(xy)^\lambda q(x)r(y)}\right)^p I^p(x,y)dx\,dy$$

$$\leq (\alpha\beta)^p\int_a^b\int_c^d (xy)^{-m+p}f^p(x,y)dx\,dy$$

$$\leq (\alpha\beta)^p\int_0^\infty\int_0^\infty (xy)^{-m+p}f^p(x,y)dx\,dy.$$

Let $a<s<b$ and $c<t<d$. Then

$$\int_a^s\int_c^t (xy)^{-m}\left(\frac{1}{(xy)^\lambda q(x)r(y)}\right)^p I^p(x,y)dx\,dy$$

$$\leq (\alpha\beta)^p\int_0^\infty\int_0^\infty (xy)^{-m+p}f^p(x,y)dx\,dy.$$

Let $b\to\infty$ and $d\to\infty$ we see that

$$\int_a^s\int_c^d (xy)^{-m}H^p(x,y)dx\,dy$$

$$\leq (\alpha\beta)^p\int_0^\infty\int_0^\infty (xy)^{-m+p}f^p(x,y)dx\,dy,$$

holds for all a,s,c,t such that $0<a<s$, $0<c<t$, the required inequality then follows.

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