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ON GENERALIZED (N, p_n, q_n) SUMMABILITY OF JACOBI AND LAGUERRE SERIES

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Abstract. In this paper we generalize two theorems of Szego [7] for Jacobi and Laguerre series by $G(N, p, q)$ summability method.

1. Notation and Definitions

The (N, p, λ) transform of $s_n = \sum\limits_{v=0}^{n} a_v$ is defined by

$$T_n = \frac{\sum\limits_{v=0}^{n} p_{n-v}\lambda_v s_v}{\gamma_n}$$

where

$$\gamma_n = \sum_{=0}^{n} p_v \lambda_{n-v} \quad (p_{-1} =_{-1}= \gamma_{-1} = 0),$$
$$\neq 0 \quad \text{for} \quad n \geq 0.$$

The series $\sum\limits_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable (N, p, λ) to s, if $T_n \to s$ as $n \to \infty$ and is said to be absolutely summable $|N, p, \lambda|$ if $\{T_n\} \in BV$ and when this happens, we shall write symbolically by $\{s_n\} \in |N, p, \lambda|$. The method (N, p, λ) reduces to method (N, p_n) when $\lambda_n = 1$ [Hardy [3] 64]; to the Euler-Knopp method (E, δ) when $p_n = \rho^n \delta^n / \underline{|n}$, $\lambda_n = \alpha^n / \underline{|n}$ $(\alpha > 0, \delta > 0)$ [(3) 178]; to the method (C, ρ, β) [Browein (1)] when $p_n = \binom{n+\rho-1}{\rho}$ $_n = \binom{n+\beta}{\beta}$.

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We write,

$$\begin{aligned}\epsilon_n &= p_n - p_{n-1} = \Delta p_n \\ \xi_n &= q_n - q_{n-1} = \Delta q_n \\ \mu_n &= \delta_n^\rho \quad \text{where} \quad \delta_n = \sum_{v=0}^{n} \lambda_v\end{aligned}$$

and δ_n^ρ is the n^{th} Cesqro mean of the sequence $\{\lambda_n\}$ of order ρ we noted that

$$\gamma_n = \sum_{v=0}^{n} p_{n-v}\lambda_v = \sum_{=0}^{n} \epsilon_{n-v}\delta_v$$

and

$$\begin{aligned}\sum_{v=0}^{n} p_{n-v}\lambda_v S_v &= \sum_{=0}^{n}(p_{n-v} - p_{n-v-1})\sum_{i=0}^{v} \lambda_i S_i \\ &= \sum_{=0}^{n} \epsilon_{n-v} t_v S_v\end{aligned}$$

Here $\{t_v\}$ is the $(\overline{N}, \lambda)$ mean [[3] p.57] which is equivalent to $(R, \delta_{n-1}, 1)$ mean [[3] 113] Rewriting T_n in terms of the simplification given above, we have

$$T_n = \frac{\sum\limits_{v=0}^{n} (p_{n-v} - p_{n-v-1}) t_v \delta_v}{\sum\limits_{v=0}^{n} (p_{n-v} - p_{n-v-1}) \delta_v}$$

and this form suggest that we can obtain the following extension of the (N, p, λ) method.

We now write for any $\{\epsilon_n\}$

$$T_n^{(\rho)} = \frac{\sum\limits_{v=0}^{n} \epsilon_{n-v} t_v^\rho \delta_v^\rho}{\sum\limits_{v=0}^{n} \epsilon_{n-v} \delta_i^\rho} = \frac{\sum\limits_{v=0}^{n} \epsilon_{n-v} t_v^\alpha \mu_v}{\sum\limits_{v=0}^{n} \epsilon_{n-v} \mu_v} \tag{1.1}$$

where

$$t_n^{(\rho)} = \frac{1}{\delta_n^\rho} \sum_{v=0}^{n} (\delta - \delta_{v-1})^\alpha a_v$$

we denoted this mean by $G(N, p, \lambda)_\rho$ [Dhal (2)]. When $\rho = 1$

$T_n^{(1)} = (N,p,\lambda)\ (S_n)$, the $G(N,p,\lambda)\rho$ method reduces to (N,p,λ) method.

2.

The Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$ $\alpha > -1$ $\beta > -1$ can be defined from the generating function [see (7)]

$$\sum_{n=0}^{\infty} p_n^{(\alpha,\beta)}(x)W^n$$
$$= 2^{\alpha+\beta}(1-2XW+W^2)^{\frac{-1}{2}}[1-w+(1-2XW+W^2)^{-\frac{1}{2}}]^{-\alpha}$$
$$[1+W+(1-2XW+W^2)^{\frac{1}{2}}]^{-\beta}$$

and the Laguerre polynomials $L_n^{(\alpha)}(X)$ $\alpha > -1$ from the generating function [(7)]

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(X)W^n = (1-W)^{-\alpha-1}\exp\left(\frac{-XW}{1-W}\right)$$

The formal expansion of a measurable function $f(X)$ defined on $[-1,1]$ in a Jacobi series is

$$f(X) \sim \sum_{n=0}^{\infty} a_n p_n^{(\alpha,\beta)}(X) \tag{1}$$

where

$$a_n = \frac{1}{h_n}\int_{-1}^{1}(1-X)^\alpha(1+X)^\beta\ f(x)p_n^{(\alpha,\beta)}(X)dx$$

$$h_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\cdot\frac{\Gamma(n+\alpha+1)}{(n+1)}\cdot\frac{\Gamma(n+\beta+1)}{(n+\alpha+\beta+1)}$$

and all the integerals are assumed to exist. The expansion of a measurable function $g(x)$ defined on $(0,\infty)$ in a Laguerre series is

$$g(x)\sum_{n=0}^{\infty} b_n L_n^{(\alpha)}(x) \tag{2}$$

where

$$b_n = \frac{}{\Gamma(\alpha+1)A_n^\alpha}\int_\theta^\infty e^{-x}x^\alpha f(x)L_n^{(\alpha)}(x)dx$$

The object of this note is to extend the result on the (C, K) summability of (1) and (2) to a wider class of generalized (N, p, q) methods. The method of proof due to Karamata (see (4)]. It's great advantage it avoids the necessity of obtaining estimates of the Kernal of the generalized (N, p, q) transformation of (1) and (2) which are more difficult to obtain in the case of Fourier series.

The following two result appear implicity in Szegö [7].

Theorem A. *Let $f(x)$ be Lebesgue measurable in $[-1, 1]$ and suppose*

$$\int_0^t |f(\cos\theta) - S|d\theta = 0(t) \tag{3}$$

as $t \to 0$. For $\alpha > -1$, $\beta > -1$ suppose the following integral exists.

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta |f(x)|dx \tag{4}$$

Then the Jacobi series (1) of $f(x)$ is (C, K) summable to s at the point $x = 1$ provided that $k > \alpha + \frac{1}{2}$ and in the $\beta > -\frac{1}{2}$, $\alpha + \frac{1}{2} < k < \alpha + \beta$ the following additional antipole condition is satisfied the integral

$$\int_{-1}^{0} (1+x)^{\frac{\beta}{2}-\frac{1}{4}} |f(x)|dx \quad \textit{exist.} \tag{5}$$

Theorem B. *Let $g(x)$ be Lebesgue measureable in $[0, \infty]$ and suppose*

$$\int_0^t |g(x) - U|dx = 0(t) \tag{6}$$

as $t \to 0$ for $\alpha > -1$ suppose the following integral exist

$$\int_{-1}^{\infty} e^{-x^2} x^{\alpha-k-\frac{1}{3}} |g(x)|dx \tag{7}$$

Then the Laguerre series (2) of $g(x)$ is (C, K) summable to U at the point $x = 0$ provided that $k > \alpha + \frac{1}{2}$.

These are based on theorem 9.1.4, 9.1.7 respectively in [7]. Theorem A is proved explicitly as remark (4) in the proof of Theorem 9.1.4 and the proof of

Theorem 9.1.7 can be altered in the same way to give Theorem B. (4) We prove the following two theorems which reduce to Theorem A and B when $p_n = A_n^{K-1}$, $q_n = 1$, $\rho = 1$.

Theorem 1. *Let p_n, $q_n \geq 0$ and p_n be non-increasing and such that*

$$\sum_{i=0}^{n} \frac{(*\mu)}{A_i^{\alpha+\frac{3}{2}}} = 0\left(\frac{(\epsilon * \mu)_n}{A_n^{\alpha+\frac{1}{2}}}\right) \tag{8}$$

holds. Then the Jacobi series (1) of $f(x)$ is $G(N,p,\lambda)_\rho$ summable to S at the point $x = 1$ provided that $-\frac{1}{2} < \alpha < \frac{1}{2}$ and (3), (4) hold.

Theorem 2. *Let $p_n, q_n \geq 0$ and $\{p_n\}$ be non-increasing and suppose (8) holds. Then the Laguerre series (2) of $g(x)$ is $G(N,p,\lambda)\rho$ summable to U at the point $x = 0$ provided that $-\frac{1}{2} < \alpha < \frac{1}{2}$ and (6) and (7) hold.*

It can be remarked that in this case $-1 < \beta \leq -\frac{1}{2}$ (5) follows from (4). The proof of these two theorem follow immediately from the following inclusion theorem and Theorem A and B.

Theorem 3. *Suppose that $-\frac{1}{2} \leq \alpha < \frac{1}{2}$, $p_n, q_n \geq 0$ $\{p_n\}$ is non-increasing and that (8) holds then there exists $\delta > 0$ such that*

$$(C, \alpha + \frac{1}{2} + \delta) \Rightarrow G(N,p,\lambda)_\rho$$

More previsely, if,

$$\lim_{n\to\infty} \frac{A_n^{\alpha+\frac{1}{2}}}{(\epsilon * \mu)_n} \sum_{i=0}^{n} \frac{(\epsilon * \mu)_i}{A_i^{\alpha+\frac{3}{2}}} = \phi < \infty \tag{9}$$

Then $(C, \alpha + \frac{1}{2} + \delta) \Rightarrow G(N,p,\lambda)_\rho$ for all δ satisfying (10)

$$0 < \delta < \min(\frac{1}{2} - \alpha, \frac{\alpha + \frac{3}{2}}{\phi}) \tag{10}$$

Proof. We first show that the restriction of the rate of the growth of the $(\epsilon * \mu)_n^{\prime S}$ ensures that $(\epsilon * \mu)_n > M_n^{\alpha+\frac{1}{2}+\delta}$ for all $0 < \delta < \alpha + \frac{3}{2}/\phi$ where M denotes a constant that may be different at each occurrence. Let C_n be defined by

$$\begin{aligned}\frac{C_n(\epsilon * \mu)_n}{A_n^{\alpha+\frac{1}{2}}} &= \sum_{i=0}^{n-1} \frac{(\epsilon * \mu)_i}{A_i^{\alpha+\frac{3}{2}}} \\ &= \sum_{i=0}^{n-2} \frac{(\epsilon * \mu)_i}{A_i^{\alpha+\frac{3}{2}}} + \frac{(\epsilon * \mu)_{n-1}}{A_{n-1}^{\alpha+\frac{3}{2}}} \\ &= \frac{C_{n-1}(\epsilon * \mu)_{n-1}}{A_n^{\alpha+\frac{3}{2}}}\left[1 + \frac{1}{C_{n-1}} \cdot \frac{\alpha+\frac{3}{2}}{\alpha+n+\frac{1}{2}}\right]\end{aligned}$$

So that

$$\frac{C_n(\epsilon * \mu)_n}{A_n^{\alpha+\frac{1}{2}}} = (\epsilon * \mu)_0 \prod_{i=2}^{n}\left[1 + \frac{\alpha+\frac{3}{2}}{(\alpha+i+\frac{1}{2})_{C_{i-1}}}\right]$$

Now (9) implies that

$$|\sup_{J \geq n} C_J - \phi| \longrightarrow 0$$

as $n \to \infty$, and therefore, if $0 < \delta < \frac{1}{\phi}(\alpha + \frac{3}{2})$, $n > 0$

$$\frac{(\epsilon * \mu)_n}{A_n^{\alpha+\frac{1}{2}}} > p_0 q_0 \frac{\delta}{\alpha+\frac{3}{2}} \prod_{i=2}^{n_0}\left[1 + \frac{\alpha+\frac{3}{2}}{(\alpha+i+\frac{1}{2})C_{i-1}}\right] \prod_{i=n_0+1}^{n}\left[1 + \frac{\delta}{\alpha+i+\frac{1}{2}}\right]$$

and therefore we have

$$(\epsilon * \mu)_n > MA_n^{\alpha+\frac{1}{2}} \prod_{i=n_0+1}^{n}\left[1 + \frac{\delta}{i+1}\right] = M_n^{\alpha+\frac{1}{2}+}$$

Let $\{k_n\}$ be defined as

$$\sum_{n=0}^{\infty} D_n X^n = \sum_{n=0}^{\infty} k_n X^n \sum_{n=0}^{\infty} A_n^{\alpha-\frac{1}{2}+\delta} X^n$$

so that

$$(\epsilon * \mu)_n = \sum_{i=0}^{n} k_{n-i} A_i^{\alpha+\frac{1}{2}+\delta}, \quad k_n = \sum_{i=0}^{n} D_{n-i} A_i^{-\alpha-\frac{3}{2}-\delta}$$

By Hardy [3] in order that $(C, \alpha + \frac{1}{2} + \delta) \Rightarrow G(N, p, \lambda)\rho$ it is necessary and sufficient that

$$\sum_{i=0}^{n} A_i^{\alpha+\frac{1}{2}+}|k_{n-i}| = 0\ (\epsilon * \mu)_n. \tag{11}$$

Now

$$\begin{aligned} k_n &= \sum_{i=0}^{n} A_i^{-\alpha-\frac{3}{2}-\delta}[D_n + (D_{n-i} - D_n)] \\ &= D_n A_n^{-\alpha-\frac{1}{2}-\delta} + \sum A_i^{-\alpha-\frac{3}{2}-\delta}(D_{n-i} - D_n) \end{aligned}$$

and assuming that (10) holds. The first term is positive and second is negative so that

$$|k_n| \leq 2D_n A_n^{-\alpha-\frac{1}{2}-\delta} - k_n$$

hence (11) would follow from

$$\sum_{i=0}^{n} A_i^{\alpha+\frac{1}{2}+\delta} A_{n-1}^{-\alpha-\frac{1}{2}-\delta} D_{n-1} = 0 \qquad (\epsilon * \mu)_n \tag{12}$$

which is turn will follow from.

$$A_n^{\alpha+\frac{1}{2}+\delta} \sum_{i=0}^{n} (-A_i^{-\alpha-\frac{1}{2}-\delta}) D_i = 0 \qquad (\epsilon * \mu)_n \tag{13}$$

since we are assuming (10) holds (13) is equivalent to

$$A_n^{\alpha+\frac{1}{2}+\delta} \sum_{i=1}^{n} (-A^{-\alpha-\frac{3}{2}-\delta})(\epsilon * \mu)_i = 0 \qquad (\epsilon * \mu)_n. \tag{14}$$

In order to prove that (14) holds, let

$$\gamma_n = \sum_{i=0}^{n} \frac{(\epsilon * \mu)_n}{A_i^{\alpha+\frac{3}{2}}}$$

and for $\theta > 0$, $n > 0$

$$\begin{aligned} \frac{\gamma_n}{A_n^{\theta}} - \frac{\gamma_{n-1}}{A_{n-1}^{\theta}} &= \frac{1}{A_n^{\theta}}(\gamma_n - \frac{(n+\theta)}{n}\gamma_{n-1}) \\ &= \frac{1}{A_n^{\theta}}\left[\frac{(n+\theta)(\epsilon * \mu)_n}{nA_n^{\alpha+\frac{3}{2}}} - \frac{\theta\gamma_n}{n}\right] \\ &= \frac{(\epsilon * \mu)_n}{nA_n^{\theta}A_n^{\alpha+\frac{1}{2}}}\left[\frac{(n+\theta)(\alpha+\frac{3}{2})}{(n+\alpha+\frac{3}{2})} - \frac{\theta\gamma_n A_n^{\alpha+\frac{1}{2}}}{(\epsilon * \mu)_n}\right] \end{aligned}$$

If θ is chosen so that $\theta_\phi < (\alpha + \frac{3}{2})$, then by (9)

$$\frac{\gamma_n}{A_n^\theta} - \frac{\gamma_{n-1}}{A_{n-1}^\theta} \geq 0 \tag{15}$$

for n sufficiently large. To prove (14) it is sufficient to show

$$A_n^{\alpha+\frac{1}{2}+} \sum_{i=1}^{n} (-A_i^{-\alpha-\frac{2}{3}-}) A_i^{\alpha+\frac{3}{2}} (\gamma_i - \gamma_{i-1}) = 0(\epsilon * \mu)_n$$

and by partial summation this is equivalent to

$$A_n^{\alpha+\frac{1}{2}+\delta} \sum_{i=1}^{n} \gamma_n \left[A_{i+1}^{-\alpha-\frac{3}{2}-\delta} A_i^{\alpha+\frac{3}{2}} - A_i^{-\alpha-\frac{3}{2}-\delta} A_i^{\alpha+\frac{3}{2}} \right] = 0(\epsilon * \mu)_n$$

For a sufficiently large fixed N, it is sufficient, using (15) to prove

$$A_n^{\alpha+\frac{1}{2}+\delta} \frac{\gamma_n}{A_n^\theta} \sum_{i=N}^{n} A_i^\theta \left[A_{i+1}^{-\alpha-\frac{3}{2}-\delta} A_{i+1}^{\alpha+\frac{3}{2}} - A_i^{-\alpha-\frac{3}{2}-\delta} A_i^{\alpha+\frac{3}{2}} \right] = 0(\epsilon * \mu)_n$$

where $0 < \theta < \frac{1}{\phi}\ (\alpha + \frac{3}{2})$. Now the term in side the sum is $0(n^\theta - \delta - 1)$ and so L. H. S. is $0(\gamma_n \cdot n^{\alpha+\frac{1}{2}})$ if $\theta > \delta$, and this is clearly $0(\epsilon * \mu)_n$ by (8). Hence (14) holds and the theorem is proved.

Remark. For $q_n = 1$. An our theorem to Thorpe [9].

Corollary. *Under the hypothesis of Theorem 3, $\phi \geq \frac{3+2\alpha}{1-2\alpha}$ and so (10) can be put in simpler form.*

$$0 < \delta < \frac{\alpha + \frac{3}{2}}{\phi}$$

Proof. Since $\{p_n\}$ is non-increasing and $q_n \geq 0$ it is easy to show that $(\epsilon * \mu)_{n/n+1}$ is non-increasing. Hence

$$\frac{(\epsilon * \mu)_i}{i+1} \geq -\frac{(\epsilon * \mu)_n}{n+1} \quad (0 \leq i \leq n+1)$$

So that

$$\begin{aligned} \frac{A_n^{\alpha+\frac{1}{2}}}{(\epsilon * \mu)_n} \sum_{i=0}^{n} \frac{(\epsilon * \mu)_i}{A_i^{\alpha+\frac{3}{2}}} &= \frac{A_n^{\alpha+\frac{1}{2}}}{(\epsilon * \mu)_n} \sum_{i=0}^{n} \frac{(\epsilon * \mu)_i (i+1)}{(i+1) A_i^{\alpha+\frac{3}{2}}} \\ &\geq \frac{A_n^{\alpha+\frac{1}{2}}}{(n+1)} \sum_{i=0}^{n} \frac{(i+1)}{A_i^{\alpha+\frac{3}{2}}} \end{aligned}$$

using the asymptotic expressions.

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