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COMPACT LIE GROUP ACTIONS ON ASPHERICAL $A_k(\pi)$-MANIFOLDS[1]

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Abstract. Let M be an aspherical $A_k(\pi)$-manifold and π' torsion-free, where π' is some quotient group of π. We prove that (1) Suppose the Euler characteristic $\chi(M) \neq 0$ and G is compact Lie group acting effectively on M, then G is finite group (2) The semisimple degree of symmetry of M $N_T^s \leq (n-k)(n-k+1)/2$. We also unity many well-known results with simpler proofs.

1. Introduction and Preliminaries

A CW complex (resp. manifold) is called an aspherical complex (resp. aspherical manifold) if its covering space is contractible. An extremely important class of aspherical complexes is the class of the Eilenberg-MacLane spaces $K(\pi, 1)$. A well-known theorem of Cartan-Hadamard showed that if a cnnected complete Riemannian n-manifold M has sectional curvature $K_M \leq 0$, then the universal covering space of M is diffeomorphic to Euclidean space R^n. Hence it is a $K(\pi, 1)$-space with $\pi = \pi_1(M)$.

We will always assume that M is a compact connected oriented n-manifold unless otherwise stated. Whether or not there exists a non-trivial continuous map $f : M^n \to K(\pi, 1)$ for some group π is an extremely important problem. For instance, Gromov and Lawson have verified that if M is a spin manifold and

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there exists a map $f : M \to T^n$ of positive degree, then M does not admit a Riemannian metric with positive scalar curvature [5].

By an aspherical $A_k(\pi)$-manifold, we mean a compact connected n-manifold M together with a continuous map $f : M \to K(\pi, 1)$ such that

$$f_* : H_k(M; Q) \to H_k(K(\pi, 1); Q)$$

is non-trivial. Equivalently, we also can define an aspherical $A_k(\pi)$-manifold by using cohomology, that is, the homomorphism

$$f^* : H^k(K(\pi, 1); Q) \to H^k(M; Q)$$

is non-trivial. Throughout this paper, we shall use these notations and the Alexander-Spanier cohomology with compact supports.

The group actions on the aspherical $A_k(\pi)$-manifold M is a very important problem in topology and geometry. In 1943 [12], Montgomery and Samelson proved that if a compact Lie group acts transitively and effectively on the n-torus $T^n = M = K(Z \oplus \cdots \oplus Z, 1)$ (n copies of Z), (where f is the identity), then $G \cong T^n$, and G acts freely on T^n. Since then this problem has been investigated by many mathematicians such as Donnely-Schultz [4], Conner-Montgomery [3], H. T. Ku and M. C. Ku [8], [9], Scheon-Yau [13], etc. In this paper, we shall prove some new results concerning compact Lie group actions on aspherical $A_k(\pi)$-manifolds. As a by product, we are able to give new simple proofs for some well-known results.

Now, let G be a compact Lie group acting on M. Define the map $i : G \to M$ by $i(x) = g(x)$ for every $g \in G$, where x is any fixed base point. Let $p : M \to M/G$ be the orbit map, and r the dimension of any principal orbit. If $f : M \to K(\pi, 1)$ is continuous, it is well known that $f_* i_* \pi_1(G) \subset$ center of π. Hence we can define the quotient group $\pi' = \pi / f_* i_* \pi_1(G)$. Let $\alpha : \pi \to \pi'$ be the quotient map.

Lemma 1.1. *Let $f : M \to K(\pi, 1)$ be a continuous map. Suppose that π' is torsion free. Then there exists a continuous map*

$$h : M/G \to K(\pi', 1),$$

such that $ph \simeq \alpha f$ *(that is,* ph *is homotopic to* αf*), where* $\alpha : K(\pi, 1) \to K(\pi', 1)$ *is induced by* $\alpha : \pi \to \pi'$*. In particular*

$$\alpha_* f_* H_i(M; L) = 0, \quad \textit{for} \quad i > n - r,$$

where L *is any integral domain.*

Proof. By [4], $\pi_1(M/G) \cong [\pi(M)/i_*\pi(G)]/N$, where N is generated by a set of elements of finite order. Since π' is torsion-free, hence $\operatorname{Ker} p_* \subset \operatorname{Ker}(\alpha f)_*$, where
$p_* : \pi(M) \to \pi_1(M/G)$. By [4], there exists a continuous map
$h : M/G \to K(\pi', 1)$ such that $ph \simeq \alpha f$. Thus,
$h_* p_* = \alpha_* f_* : H_*(M; L) \to H_*(K(\pi', 1); L)$.

As $H_i(M/G; L) = 0$ for $i > n - r$, because $\dim M/G = n - r$. The result follows. Observe that the conclusion of Lemma 1.1 remains true if π is torsion free. Then we have a map $h : M/G \to K(\pi, 1)$ such that $ph \simeq f$ and $f_* H_i(M; L) = 0$ for $i > n - r$.

Corollary 1.2. *Under the hypotheses of Lemma 1.1, if* $f_* i_* \pi_1(G)$ *is finite, then*

$$f_* H_i(M; Q) = 0, \quad \textit{for} \quad i > n - r$$

In particular, this conclusion always holds if the action of G *has a fixed point.*

Proof. We have a fibre space

$$K(f_* i_* \pi_1(G), 1) \to K(\pi, 1) \xrightarrow{\alpha} K(\pi', 1).$$

Since $f_* i_* \pi_1(G)$ is a finite group, $H_i(K(f_* i_* \pi_1(G), 1); Q) = 0$ for $i > 0$. Hence by the Vietoris-Begle mapping theorem, we have isomorphisms

$$\alpha_* : H_i(K(\pi, 1); Q) \simeq H_i(K(\pi', 1); Q), \quad \text{for} \quad i \geq 0$$

This will imply the desired result by Lemma 1.1.

2. Toral Group Actions on Aspherical $A_k(\pi)$-manifolds

Let $G = T^l$ be a toral group acting on a compact Poincaré duality space or a rational cohomology n-manifold M. Let E_G denote the universal space of G and $M_G = E_G \times_G M$. Let $i : M \to M_G$ be the inclusion and $p : M \to M/G$ be the orbit projection as above. Following [11], for $z \in H^s(M/G; Q)$ and $w \in Im\{i^* : H^{n-s}(M_G; Q) \to H^{n-s}(M; Q)\}$, the rational number

$$p(w, x) = \langle w \cup p^*(z), [M] \rangle$$

is called a characteristic number of the orbit map, where $[M]$ denotes the fundamental homology class of M. If M is a smooth manifold, w is a product of Pontrjagin classes of M, and the action is smooth, then $p(w, z)$ is simply the usual Pontrjagin number of the orbit map.

Lemma 2.1. (Ku-Ku [10]). *Let $G = T^l$ act on a compact Poincaré duality n-space or a rational cohomology n-manifold M with finite robit types. Suppose $p(w, z) \neq 0$ for some w and $z \in H^s(M/G; Q)$. Then $H^s(F; Q) \neq 0$, where $F = F(G, M)$ denotes the fixed point set.*

Let $\pi' = \pi / f_* i_* \pi_1(G)$ as above.

Theorem 2.2. *Let M be an aspherical $A_k(\pi)$-manifold and π' torsion-free. Suppose that the Euler Characteristic $\chi(M)$ is non-zero, or $p(w, z) \neq 0$ for some w and z. Then*

(1) $k \leq n - r$;

(2) *There is a component F_0 of the fixed point set $F = F(G; M)$ which is an aspherical cohomology $A_k(\pi)$-manifold.*

Proof. The fixed point set F is not empty because $\chi(F) = \chi(M) \neq 0$, or by Lemma 2.1 because $p(w, z) \neq 0$. Thus, from Corollary 1.2 we have

$$f_* H_i(M; Q) = 0, \quad \text{for} \quad i > n - r.$$

But $f_* : H_k(M; Q) \to H_K(K(\pi, 1); Q)$ is non-trivial because M is an aspherical $A_K(\pi)$-manifold. Therefore, $K \leq n - r$. This proves (1).

To prove (2), let $F = \cup_{j=1}^{s} F_j$, where F_j's are components of F, and let $f_j = f|F_j : F_j \to K(\pi, 1)$. By Smith theory, each component F_j is a cohomology manifold. Suppose that

$$F_j^* : H^k(K(\pi,1);Q) \to H^k(F_j;Q)$$

is trivial for every $j = 1, 2, \cdots, m$. We shall proceed to get a contradiction. From Lemma 1.1, there is a map $h : M/G \to K(\pi', 1)$ such that $hp \simeq \alpha f$. Let $u : F \to M$ and $v : F \to M/G$ be inclusions. Then

$$v^*h^* = (hv)^* = (hpu)^* = (\alpha f u)^* = \sum_{j=1}^{m} f_j^* \alpha^* = 0$$

in degree k. Since M is an aspherical $A_k(\pi)$-manifold and $\alpha^* : H^k(K(\pi',1);Q) \to H^k(K(\pi,1);Q)$ is an isomorphism, hence

$$f^*\alpha^* : H^k(K(\pi',1);Q) \to H^k(M;Q)$$

is non-trivial. Let $\overline{y} \in H^k(K(\pi,1);Q)$ be such that $f^*\alpha^*(\overline{y}) = y \neq 0$. Since $v^*h^*(\overline{y}) = 0$, from the exact cohomology sequence of the pair $(M/G, F)$, there exists an element $y' \in H^*((M-F)/G;Q)$ such that $j'^*(y') = h^*(y)$, where j' is an inclusion. We have commutative diagram:

$$\begin{array}{ccccc}
H_G^k(F;Q) & \xleftarrow{\Sigma i_j^*} & H_G^k(M;Q) & \xleftarrow{\quad j^* \quad} & H_G^k(M-F;Q) \\
 & & {\scriptstyle i^*}\big\downarrow \quad \nwarrow & & \big\uparrow{\scriptstyle \overline{p}^*} \\
 & & H^k(M;Q) \xleftarrow{p^*} H^k(M/G;Q) & \xleftarrow{j'^*} & H^*(M-F;q)
\end{array}$$

where j is the inclusion, and $\overline{p} = p|(M-F)$. Thus we have

$$\sum_{j=1}^{m} i_j^*(j^*\overline{p}^*y') = 0, \tag{1}$$

because the top sequence above is exact, and

$$i^*(j^*\overline{p}^*y') = p^*j'^*y' = p^*h^*(\overline{y}) = f^*\alpha^*(\overline{y}) = y.$$

Hence $j^*\overline{p}^*y'$ is $H^*(B_G;Q)$-free. But by the Borel Localization Theorem [14] we have

$$\sum_{j=1}^{m} S^{-1}i_j^* : S^{-1}i^*H_G^*(M;Q) \simeq S^{-1}H_G^*(F;Q),$$

where $S = H^*(B_G;Q) - -\{0\}$. Hence $S^{-1}\Sigma i^*[j^*\overline{p}y'] \neq 0$, where $[j^*\overline{p}y']$ denotes the class of $j^*\overline{p}y'$ in $S^{-1}H_G^*(M;Q)$. But $S^{-1}\Sigma i_j^*[j^*\overline{p}y'] = 0$ by (1). This is a contradiction. Hence there exists at least one component F_0 of F such that

$$(f|F_0)^* : H^k(K(\pi,1);Q) \to H^k(F_0;Q)$$

is non-trivial. Therefore F_0 is an aspherical cohomology $A_k(\pi)$-manifold. If π is torsion free and $\chi(M) \neq 0$, a similar result is proved in [6].

Theorem 2.3. *Let M be an aspherical $A_k(\pi)$-manifold and π' is torsion free. Suppose that $\chi(M) \neq 0$, or $p(w,z) \neq 0$ for some w and z. If G is a compact Lie group acting effectively on M, then G is finite group.*

Proof. Suppose G is not finite, then G contains a toral subgroup T^s, $s \geq 1$, and T^s still acts effectively on M. By hypothese, the action of T^s on M has non-empty fixed point set. Hence by Corollary 1.2, $f_*H_i(M;Q) = 0$ for $i > n - r$. On the other hand, $f_* : H_n(M;Q) \to H_n(K(\pi,1);Q)$ is non-trivial. This is contradiction because $r \geq 1$.

As an easy corollary we have the following result. The case $\chi(M) \neq 0$ is the main theorem of Conner and Montgomery in [3].

Theorem 2.4. *Let G be a compact Lie group acting effectively on a closed connected aspherical manifold M with $\chi(M) \neq 0$, or $p(w,z) \neq 0$ for some w and z. Then G is finite.*

If we define the degree of symmetry $N_T(M)$ of M as the supremum of the dimensions of all Lie groups which can act effectively on M. Then we have the following result from Theorem 2.3 and 2.4.

Theorem 2.5. *Let M be an aspherical $A_k(\pi)$-manifold and π' is torsion* *free. Suppose that $\chi(M) \neq 0$. Then $N_T(M) = 0$. In particular, if M is an aspherical manifold with $\chi(M) \neq 0$, $N_T(M) = 0$.*

3. Semi-Simple Degree of Symmetry

Suppose that M is a compact connected n-manifold. The semi-simple degree of symmetry $N_T^s(M)$ (resp. $N^s(M)$) of M is defined as the supremum of the dimensions of all compact semi-simple Lie groups which can act effectively (resp. effectively and smoothly) on M. The semi-simple degree of symmetry has an interesting connection with the Riemannian geometry, that is, if $N^s(M) \neq 0$, then M admits a Riemannian metric with positive scalar curvature [11].

Theorem 3.1. *Let M be an aspherical $A_k(\pi)$-manifold. If π' is torsion-free. Then $k \leq n - r$. Hence*

$$N_T^s(M) \leq (n-k)(n-k+1)/2.$$

Proof. Let G be a compact semi-simple Lie group acting effectively on M with $\dim G = N_t^s(M)$. Since $\pi_1(G)$ is finite, so is $f_* i_* \pi_1(G)$. Hence $f_* H_i(M; Q) = 0$ for $i > n - r$. But $f_* H_k(M; Q) \neq 0$, hence $r \leq n - k$. Let $G(x)$ be a principal orbit. Then $\dim G(x) = r \leq n - k$. But the group acts effectively on $G(x)$, hence

$$\dim G \leq (n-k)(n-k+1)/2.$$

This gives the desired conclusion.

Let $N = K(\pi, 1)$. If N is a Riemannian n-manifold of negative curvature, and M, f are both smooth, Theorem 3.1 in this special case was proved by Scheon and Yau in [13]. In [4], Donnelly and Schultz remove the smoothness condition. The general case with π torsion-free was proved in [9]. A similar result for paracompact space M and π torsion-free was proved by Berstein in [1].

Theorem 3.2. *Let M be an asperical $A_k(\pi)$-manifold with $N_T^s(M) = 0$. Suppose π' is torsion-free, $N_T(M) \neq 0$ and $\chi(M) \neq 0$. Then*

$$N_T(M) \leq \min(n-k, [n/2])$$

Proof. Since $N_T^s = 0$, let T^s act effectively on M, where $s = N_T(M)$. Since $\chi(M) \neq 0$, $F(T^s, M) \neq \emptyset$. Again, by using Corollary 1.2 we can verify that $k \leq n-r$. But for an effective toral group action, the principal isotopy subgroups are finite, hence $r = s$. It follows that $s \leq n-k$. Since $F \neq \emptyset$, from a result of H. T. Ku [6], we also have $s \leq [n/2]$. This proves the theorem. A similar result was also proved in [9] under the following hypotheses: M aspherical $A_k(\pi)$-manifold, $N_T^s(M) = 0$, $N_T(M) \neq 0$, $\chi(M) \neq 0$, $\pi_1(M)$ abelian and π torsion-free.

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