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GEODESIC TUBES ON LOCALLY SYMMETRIC SPACES

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Abstract. In this paper we state and prove a characteristic relation which exists, between the eigenspaces of the Ricci transformation $R(N,-)N$ acting on the orthocomplement space of N in T_mM where $m \in M$, M being a locally symmetric space, and the Weingarten map S_N of small enough geodesic tubes of M.

1. Introduction

It is an interesting problem to see how the properties of geodesic tubes on a Riemannian manifold (M,g) determine the geometry of the ambient space (M,g). This problem has been recently treated in [12] and [13] with (M,g) being a space of constant curvature and a space of constant holomorphic sectional curvature respectively. Similar problems have also been studied earlier, by using the properties of small geodesic spheres, in several papers, see for example [1], [7], [8], [9], [10], [11]. The proofs of the latest papers are based on a significant relation which exists, between the eigenspaces of the Ricci transformation $R(N,-)N$ acting on $\{N\}^{\perp}$ and the Weingarten map S_N of small enough geodesic spheres of the space, where N is a unit tangent vector field along a geodesic starting from the centre of the sphere.

In this aspect, we give a corresponding result for geodesic tubes of a locally

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symmetric space, by using Fermi coordinates, Fermi vector fields and a nice relation which exists between them.

In the second section we give the definitions of Fermi coordinates and Fermi vector fields, some basic properties and the definition of the tubes and tubular hypersurfaces of a Riemannian manifold. In the third section we state and prove our main result in Theorem 3.1.

2. Preliminaries

Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold, ∇ its Riemannian connection and

$$R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{2.1}$$

its curvature transformation.

Let $\sigma : (a, b) \to M$; $a, b \in \mathbb{R}$ be a curve of finite length of M. To describe the geometry of a Riemannian manifold M in the neighbourhood of a curve σ we use Fermi coordinates. E. Fermi introduced these coordinates in [3] and soon utilized by Levi-Civita and L. Eisenhart in the 1920's. We give now some useful definitions and properties, following closely [5] and [6]. To define a system of Fermi coordinates we need, an open neighbourhood of $U = U(\sigma)$ of σ, for which every point of U can be joined to σ by a shortest unit-speed geodesic, meeting σ orthogonally and it is assumed that it contains no focal points of σ. Further, we need an orthonormal frame field $\{E_1, \ldots, E_n\}$ along the curve σ, which may be a geodesic of M. Let $m = \sigma(0)$ and $\dot\sigma(t) = E_1|_{\sigma(t)}$.

Definition 2.1. The *Fermi coordinates* $(x_1, \ldots, x_n)$ of U centered at $m = \sigma(0)$, relative to a given orthonormal frame field $\{E_1, \ldots, E_n\}$, along the curve σ for which $\dot\sigma(t) = E_1|_{\sigma(t)}$, are the real-valued functions defined by

$$x_1\left(\exp_{\sigma(t)} \sum_{j=2}^{n} t_j E_j|_{\sigma(t)}\right) = t \tag{2.2}$$

$$x_i\left(\exp_{\sigma(t)}\sum_{j=2}^{n}t_jE_j|_{\sigma(t)}\right) = t_i, \quad 2\le i\le n \tag{2.3}$$

provided that the numbers $t_2,\ldots,t_n$ are small enough so that $\exp_{\sigma(t)}$ to be a diffeomorphism.

Since $\exp_{\sigma(t)}$ is a diffeomorphism on U the equations (2.2) and (2.3) define a coordinate system near m. Let $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\}$ be the coordinate vector fields associated with the Fermi coordinate system $(x_1,\ldots,x_n)$.

Lemma 2.1. *If $(x_1,\ldots,x_n)$ is a system of Fermi coordinates centered at $m\in\sigma$, then the restrictions to σ of the coordinate vector fields*

$$\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_n} \tag{2.4}$$

are orthonormal.

Lemma 2.2. *Let γ be a unit-speed geodesic of M normal to σ with $\gamma(0)=m=\sigma(0)$ and let $u=\gamma'(0)$. Then there is a system of Fermi coordinates $(x_1,\ldots,x_n)$ such that for small s we have:*

$$\left(\frac{\partial}{\partial x_2}\right)_{\gamma(s)} = \gamma'(s) \tag{2.5}$$

and

$$\left(\frac{\partial}{\partial x_1}\right)_m = (\dot\sigma(t))_m, \quad \left(\frac{\partial}{\partial x_i}\right)_m = [\dot\sigma(t)]_m^{\perp}, \quad i=2,\cdots,n. \tag{2.6}$$

Furthermore,

$$(x_\alpha\circ\gamma)(s) = s\delta_\alpha^2 \tag{2.7}$$

for $1\le\alpha\le n$ where δ is the Kronecker symbol.

For proofs of these lemmas see for example [5].

Let $\mathcal{X}(U)$ be the Lie algebra of C^∞ vector fields on U. We introduce a certain finite dimensional Abelian subalgebra of the infinite dimensional Lie algebra $\mathcal{X}(U)$.

Definition 2.2. Let $(x_1, \dots, x_n)$ be a Fermi coordinate system of $U = U(\sigma)$ relative to the orthonormal frame field $\{E_1, \dots, E_n\}$. We say that $X \in \mathcal{X}(U)$ is a *Fermi vector* field relative to $(x_1, \dots, x_n)$ provided

$$X = \sum_{i=2}^{n} c_i \frac{\partial}{\partial x_i} \tag{2.8}$$

where the c_i's are constants.

We define now two other simple but basic objects, s and N in terms of Fermi coordinates, since they will be needed in the following.

Definition 2.3. Let $(x_1, \dots, x_n)$ be a system of Fermi coordinates for $U = U(\sigma)$. For $s > 0$ we put

$$s^2 = \sum_{i=2}^{n} x_i^2 \quad \text{and } N = \sum_{i=2}^{n} \frac{x_i}{s} \cdot \frac{\partial}{\partial x_i}. \tag{2.9}$$

For $m \in \sigma$ it is easily proved that the definitions of s and N are independent of the choice of Fermi coordinates at m. In fact for $m' \in M$ near σ, $s(m') = d(m', \sigma)$ where d is the distance function of M. Furthermore,

$$N_{\gamma(s)} = \left(\frac{\partial}{\partial x_2}\right)_{\gamma(s)} = \gamma'(s), \quad s > 0 \tag{2.10}$$

where γ is the unique geodesic from m' to σ which meets σ orthogonally at $m = \gamma(0)$.

In what follows we assume that σ is also a geodesic of M and put $A = \frac{\partial}{\partial x_1}$. The most importance properties of s and N and Fermi vector fields are included in the following:

Lemma 2.3. [5]. *Let X be a Fermi vector field for $U = U(\sigma)$ and A, N, s as previously. Then we have:*

1. $\nabla_N N = 0$	*4.* $A(s) = 0$	*7.* $[N, sX] = X(s)N$
2. $g(N, N) = 1$	*5.* $[X, A] = [N, A] = 0$	*8.* $\nabla_N^2 Z = R(N, Z)N$
3. $N(s) = 1$	*6.* $[N, X] = -\frac{1}{s}X + \frac{1}{s}X(s)N$	

for any $Z = A + sX$.

If γ *is curve and* Y *is a vector field along* γ, *write* $Y' = \nabla_{\gamma'} Y$ *and* $Y'' = \nabla^2_{\gamma'} Y$. *Then we have,*

Definition 2.4. A vector field Y along a geodesic γ is called a *Jacobi field* if it satisfies the following second order differential equation:

$$Y'' = R(\gamma', Y)\gamma' \tag{2.11}$$

It is now understood that the vector field Z, in relation (8) of Lemma 2.3, is a Jacobi vector field. Moreover, if γ is a geodesic normal to σ at $m = \sigma(0)$ and X is a Fermi vector field on $U = U(\sigma)$, then the restrictions to γ of A and sX are also Jacobi vector fields, as one easily concludes it from the relation (8) of Lemma 2.3.

Therefore, we come to the following.

Lemma 2.4. *Let* $X_i^* = (\frac{\partial}{\partial x_i})$, $i = 1; 3, \cdots, n$ *be the coordinate Fermi fields relative to the Fermi coordinate system* $(x_1, \ldots, x_n)$ *of* $U = U(\sigma)$, *then the fields*

$$\begin{aligned} Y_1(s) &= X_1^*(s) = \left(\frac{\partial}{\partial x_1}\right)_{\gamma(s)}, \\ Y_3(s) &= sX_3^*(s) = s\left(\frac{\partial}{\partial x_3}\right)_{\gamma(s)}, \ldots, Y_n(s) = sX_n^*(s) = s\left(\frac{\partial}{\partial x_n}\right)_{\gamma(s)} \end{aligned} \tag{2.12}$$

are Jacobi vector fields along the geodesic γ.

Let $\sigma : (a, b) \to M$ be a curve of finite length (σ may also be a geodesic), of a Riemannian manifold (M, g). We give now the definition of a tube about the curve σ.

Definition 2.5. A (solid) tube of radius $s \geq 0$ about a curve σ is the set of points of M given by

$$T(\sigma, s) = \{\exp_{\sigma(t)} X | X \in M_{\sigma(t)}, g(X, X) = 1, g(X, \dot{\sigma}(t)) = 0, a < t < b\} \tag{2.13}$$

where $M_{\sigma(t)}$ denotes the tangent space of M at the point $\sigma(t)$.

For small $s > 0$, we call the hypersurface of the form

$$P_s = \{m' \in T(\sigma, s)/d(m', \sigma) = s\} \tag{2.14}$$

the *tubular hypersurface* at distance s form σ, or just *Tube*. If σ is a geodesic of M, then the corresponding tubes are called *geodesic tubes*. The vector field N now is the unit normal to each of the tubular hypersurfaces s =const., about the curve σ of M.

3. Geodesic Tubes and the Main Result

Let (M, g) be a Riemannian manifold of dimension $n > 2$ and let U be an open neighbourhood of a point m in M. Let σ be a geodesic of M and P_s the geodesic tube of radius s about σ contained in the open neighbourhood U. We always assume that the radius s of P_s is less than the distance of σ to its nearest focal point. Assume that $m = \sigma(0)$. Let $m' \in P_s$ and $\gamma = \gamma(s)$ be the geodesic of M containing m' belonging in U and meeting σ orthogonally at $m = \sigma(0)$. Suppose that $\gamma : (-r, r) \to U$, is parametrized by its arc length and $\gamma(0) = m$. Choose an orthonormal basis $\{E_1, \ldots, E_n\}$ for the tangent space M_m such that $E_1 = \sigma'(0)$, $E_2 = \gamma'(0)$ and let $\{x_i\}$, $i = 1, \cdots, n$ be the corresponding Fermi coordinate system on P_s. The unit tangent vector field to geodesic rays from m on $U - \sigma$ is then N given by (2.9), where s denotes the geodesic distance from m. Choose the nonzero vector $W_m = \sum_{i=3}^{n} a_i(\partial/\partial x_i)_m$ normal to $E_2 = N_m$. Let $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = \sum_{i=3}^{n} a_i s(\partial/\partial x_i)$ on U.

Lemma 3.1. *On $\gamma - \{m\}$, we have:*

$$\text{i) } [X_i, N] = 0, \quad \text{ii) } R(N, X_i)N = \nabla_N^2 X_i, \quad i = 1, 2 \tag{3.1}$$

Proof. The proof of (ii) is a consequence of Lemma 2.3, so we will only prove that $[X_2, N] = 0$. By Lemma 2.3 relation (7) we get

$$[N, X_2] = [N, \sum_{i=3}^{n} a_i s(\partial/\partial x_i)] = \left(\sum_{i=3}^{n} a_i \left(\frac{\partial s}{\partial x_i}\right)\right) N,$$

but since

$$s^2 = \sum_{i=2}^{n} x_i^2$$

we get

$$\frac{\partial s}{\partial x_i} = \frac{x_i}{s}$$

Therefore,

$$[N, X_2] = \left(\sum_{i=3}^{n} \frac{a_i x_i}{s} \right) N$$

But on $\gamma - \{m\}$, by the definition of Fermi coordinates $\sum_{i=3}^{n} a_i x_i = 0$. Hence we proved that

$$[N, X_2] = 0.$$

From this Lemma one immediately concludes that

$$\nabla_{X_i} N = \nabla_N X_i, \quad i = 1,2 \tag{3.2}$$

on $\gamma - \{m\}$.

Consider, now, the vector fields Y_i, $i = 1,2$ on γ defined by

$$Y_1|_{\gamma(\sigma)} = \left(\frac{\partial}{\partial x_1}\right)_{\gamma(\sigma)}, \quad Y_2|_{\gamma(\sigma)} = \sum_{i=3}^{n} a_i \sigma \left(\frac{\partial}{\partial x_i}\right)_{\gamma(\sigma)}, \quad -r < \sigma < r. \tag{3.3}$$

As a consequence of Lemma 3.1 we will have:

$$\nabla_{Y_i} N = \nabla_N Y_i, \quad i = 1,2 \tag{3.4}$$

on $\gamma - \{m\}$, and, by continuity,

$$R(N, Y_i)N = \nabla_N^2 Y_i \quad \text{on} \quad \gamma, \quad i = 1,2$$

where $N = e_2$ and $\{e_1, \ldots, e_n\}$ is the parallel translation of $\{E_1, \ldots, E_n\}$ along γ.

From the above analysis it is clear that Y_i, $i = 1,2$ are Jacobi vector fields along γ for which

$$\text{(I)} \begin{cases} Y_1(0) = E_1 \\ Y_1'(0) = 0 \end{cases}, \qquad \text{(II)} \begin{cases} Y_2(0) = 0 \\ Y_2'(0) = W_m \end{cases} \tag{3.5}$$

In particular Y_i, $i = 1,2$ are normal to $e_2 = N$ and for any point q on γ the subspace of M_q normal to $(e_2)_q$ is formed by evaluation all such Jacobi vector fields at q.

Write now $S_N = -\nabla N$. For any geodesic tube P_s in U about σ, the restriction of S_N to tangent vectors to P_s is just the Weingarten map with respect to N as unit normal vector field.

Proposition 3.1. *We have:*

$$\text{i) } S_N Y_i = -\nabla_N Y_i \tag{3.6}$$

$$\text{ii) } R(N, Y_i)N = S_N^2 Y_i - (\nabla_N S_N)Y_i, \quad i = 1,2 \tag{3.7}$$

for all vector fields Y_i, $i = 1,2$ orthogonal to N, along γ.

Proof.

i) From the definition of the Weingarten map (shape operator) and the relation (3.4) we have:

$$S_N Y_i = -\nabla_{Y_i} N = -\nabla_N Y_i, \quad i = 1,2.$$

ii) First, let Y_i be a Jacobi vector field as above, then along $\gamma - \{m\}$ we have:

$$\begin{aligned} R(N, Y_i)N &= \nabla_N \nabla_{Y_i} N = -\nabla_N (S_N Y_i) = -S_N \nabla_N Y_i - (\nabla_N S_N) Y_i \\ &= -S_N \nabla_{Y_i} N - (\nabla_N S_N) Y_i = S_N^2 Y_i - (\nabla_N S_N) Y_i. \end{aligned}$$

Now, since $R(N, -)N$, S_N^2 and $\nabla_N S_N$ are tensorial, from the above remarks it is valid for arbitrary vector fields Y on $U - \sigma$, where we note from the definition of S_N that

$$S_N N = -\nabla_N N = 0 \tag{3.8}$$

By continuity now the result is valid at m.

We state and prove now the main result.

Theorem 3.1. *Let m be a point in a Riemannian loally symmetric space M of dimension $n > 2$. Then m has a neighbourhood U such that, for each* *unit vector $N_m \in M_m$ and corresponding geodesic γ, the parallel translate of an eigenspace of the linear map $R(N_m, -)N_m$ along γ, is contained in an eigenspace of the Weingarten map S_N, for each geodesic tube in U about a given geodesic σ, passing from m and meeting γ orthogonally at m.*

Proof. Let M be a Riemannian locally symmetric space and $m \in M$. Following the same notation, as previously, suppose that E_1 and W_m satisfy

$$R(N_m, E_1)N_m = kE_1 \text{ and } R(N_m, W_m) = kW_m, \; k \in R. \tag{3.9}$$

Since M is locally symmetric $\nabla R = 0$, or $R(N, -)N$ is parallel along γ, hence

$$R\left(N, \frac{\partial}{\partial x_1}\right) N = k\frac{\partial}{\partial x_1} \quad \text{and} \quad R(N, W)N = kW \tag{3.10}$$

Let Y_i, $i = 1, 2$ be the Jacobi vector fields on γ satisfying (3.5).

Consider the vector field $f_1 \frac{\partial}{\partial x_1}$. We are interested for those functions f_1 for which this vector field is a Jacobi vector field along γ, with the same initial conditions (3.5-I) as Y_1. So, we will have:

$$\begin{aligned} \left(f_1 \frac{\partial}{\partial x_1}\right)_m &= E_1, \quad \text{or} \quad f_1(0) = 1 \\ \left(f_1 \frac{\partial}{\partial x_1}\right)'_m &= 0, \quad \text{or} \quad f_1'(0) = 0. \end{aligned} \tag{3.11}$$

Moreover,

$$R\left(N, f_1 \frac{\partial}{\partial x_1}\right) N = f_1 R\left(N, \frac{\partial}{\partial x_1}\right) N = kf_1 \frac{\partial}{\partial x_1}$$

and

$$\nabla_N^2 \left(f_1 \frac{\partial}{\partial x_1}\right) = f_1'' \frac{\partial}{\partial x_1}$$

Hence,

$$f_1'' = k \cdot f_1. \tag{3.12}$$

with initial conditions (3.11).

It is now easy, this equation to be solved explicitly and get

$$f_1(\sigma) = \begin{cases} \cos h(\sqrt{k}\sigma), & \text{if} \quad k > 0 \\ \cos(\sqrt{|k|}\sigma), & \text{if} \quad k < 0 \\ 1, & \text{if} \quad k = 0 \end{cases} \tag{3.13}$$

Therefore, when $\frac{\partial}{\partial x_1}$ is an eigenvector field of $R(N,-)N$, corresponding to the eigenvalue k, then $f_1 \cdot \frac{\partial}{\partial x_1}$ is a Jacobi vector field on γ when we choose f_1 given by (3.13).

Next, we are interested for those functions f_2, for which $f_2 W$ is also a Jacobi vector field along γ, with the same initial conditions (3.5-II), as Y_2. So,

$$\begin{array}{lll} (f_2 W)_m = 0 & \text{or} & f_2(0) = 0 \\ (f_2 W)'_m = W_m & \text{or} & f_2'(0) = 1 \end{array} \tag{3.14}$$

But, as previously, we have:

$$R(N, f_2 W)N = k f_2 W \quad \text{and} \quad \nabla^2_N (f_2 W) = f_2'' W \tag{3.15}$$

from which

$$f_2'' = k f_2, \tag{3.16}$$

with initial conditions (3.14). So, we have:

$$f_2(\sigma) = \begin{cases} \dfrac{1}{\sqrt{k}} \sin h(\sqrt{k}\sigma), & \text{if} \quad k > 0 \\ \dfrac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}\sigma), & \text{if} \quad k < 0 \\ \sigma, & \text{if} \quad k = 0 \end{cases} \tag{3.17}$$

Thus, when W is an eigenvector field of $R(N,-)N$ corresponding to the eigenvalue k, then $f_2 W$ is a Jacobi vector field on γ when we choose f_2 given by (3.17).

Therefore, we found that

$$Y_1 = f_1 \frac{\partial}{\partial x_1}, \quad Y_2 = f_2 W. \tag{3.18}$$

As a consequence now of (3.4) and the definition of S_N we have:

$$S_N\left(\frac{\partial}{\partial x_1}\right) = -\nabla_{\frac{\partial}{\partial x_1}} N = -\nabla_{\frac{1}{f_1} Y_1} N = -\frac{1}{f_1} \cdot \nabla_{Y_1} N = -\frac{1}{f_1} \nabla_N Y_1.$$

But

$$\nabla_N Y_1 = \nabla_N \left(f_1 \frac{\partial}{\partial x_1}\right) = (\nabla_N f_1) \frac{\partial}{\partial x_1} = N(f_1) \frac{\partial}{\partial x_1}$$

so,

$$S_N\left(\frac{\partial}{\partial x_1}\right) = -\frac{N(f_1)}{f_1}\cdot\frac{\partial}{\partial x_1} \tag{3.19}$$

or, using (3.13) we get equivalently

$$S_N\left(\frac{\partial}{\partial x_1}\right) = \begin{cases} -\sqrt{k}(\tan h\sqrt{k}\sigma)\frac{\partial}{\partial x_1}, & \text{if} \quad k>0 \\ \sqrt{|k|}(\tan\sqrt{|k|}\sigma)\frac{\partial}{\partial x_1}, & \text{if} \quad k<0 \\ 0, & \text{if} \quad k=0 \end{cases} \tag{3.20}$$

Similarly, we have

$$\begin{aligned} S_N W &= -\nabla_W N = -\nabla_{\frac{1}{f_2}Y_2}N = -\frac{1}{f_2}\nabla_{Y_2}N = -\frac{1}{f_2}\nabla_N Y_2 \\ &= -\frac{1}{f_2}\cdot\nabla_N(f_2 W) = -\frac{1}{f_2}N(f_2)W \end{aligned}$$

so,

$$S_N W = -\frac{N(f_2)}{f_2}\cdot W \tag{3.21}$$

or using (3.17) we equivalently get:

$$S_N W = \begin{cases} -\sqrt{k}(\cot h\sqrt{k}\sigma)W, & \text{if} \quad k>0 \\ -\sqrt{|k|}(\cot\sqrt{|k|}\sigma)W, & \text{if} \quad k<0 \\ -\frac{1}{\sigma}W, & \text{if} \quad k=0 \end{cases} \tag{3.22}$$

Since now the sectional curvature at m is bounded, the set of eigenvalues k of $R(N_m,-)N_m$ taken over all unit vectors N_m is bounded, say $|k| < \lambda^2$, $\lambda > 0$. Thus if U is a tubular neighbourhood of radius $< \pi/\lambda$, then f_i, $i = 1,2$ is nowhere zero on $\gamma - \{m\}$. Therefore, as a consequence, we get the required result from (3.19) and (3.21) or equivalently, from (3.20) and (3.22).

In [12] and [13] we recently characterized spaces of constant sectional curvature by using the shape operator of small enough geodesic tubes.

It seems to me now, after this result, that locally symmetric spaces of higher rank, may be also characterized, by using the properties of small geodesic tubes.

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