TAMKANG JOURNAL OF MATHEMATICS
Volume 24, Number 4, Winter 1993

A GENERALIZATION OF THE DISCRETE HARDY'S INEQUALITY

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In this note we are concerned with a generalization of the well known Hardy's inequality for series [1, pp. 239-241] (in the sequel, p shall denote a real number greater than 1):

Theorem 1. *If $p>1$, $a_k \geq 0$ for $k=1,2,3,\cdots$ and $A_k = a_1+a_2+a_3+\ldots+a_k$, then*

$$\sum_{k=1}^{n}\left[\frac{A_k}{k}\right]^p < \left[\frac{p}{p-1}\right]^p \sum_{k=1}^{n} a_k^p, \tag{1}$$

unless $a_k = 0$ for $k=1,2,\cdots,n$; and suppose further that

$$\sum_{k=1}^{\infty} a_k^p < \infty,$$

then

$$\sum_{k=1}^{\infty}\left[\frac{A_k}{k}\right]^p < \left[\frac{p}{p-1}\right]^p \sum_{k=1}^{\infty} a_k^p, \tag{2}$$

unless $a_k = 0$ for $k=1,2,3,\cdots$. The constant $(p/(p-1))^p$ is the best possible.

We shall view inequalities (1) and (2) as necessary conditions for the existence of positive nondecreasing solutions of a nonlinear recurrence relation. To be more precise, consider the following recurrence relation of the form

$$\Delta(r_{k-1}(\Delta y_{k-1})^{p-1}) + s_{k-1} y_k^{p-1} \leq 0,, \quad k=1,2,\cdots,n, \tag{3}$$

Received October 15, 1992.

where $p > 1$, $r_k > 0$ for $k = 0, 1, \cdots, n$ and $s_k \in R$ for $k = 0, 1, \cdots, n-1$. A solution of (3) is a real sequence $\{y_k\}_{k=0}^{n+1}$ which satisfies (3).

Given a fixed pair of real numbers $\alpha \geq -1$ and $\beta \leq -1$, a real sequene $y = \{y_k\}_{k=0}^{n+1}$ is said to be admissible if it satisfies the following conditions:

$$y_0 + \alpha y_1 = 0, \tag{4}$$

$$y_{n+1} + \beta y_n = 0, \tag{5}$$

$$y_k \geq 0 \quad \text{for} \quad 1 \leq k \leq n, \tag{6}$$

$$\Delta y_k \geq 0 \quad \text{for} \quad 1 \leq k \leq n-1. \tag{7}$$

Note that the condition $\alpha \geq -1$ implies that $\Delta y_0 \geq 0$ and the condition $\beta \leq -1$ implies that $\Delta y_n \geq 0$.

For any admissible sequence $v = \{v_k\}_{k=0}^{n+1}$, define the functional

$$J[v] = r_0(1+\alpha)^{p-1} v_1^p + \sum_{k=1}^{n-1} r_k (\Delta v_k)^p - \sum_{k=1}^{n} s_{k-1} v_k^p - r_n(-1-\beta)^{p-1} v_n^p.$$

We shall need the following well known result,

Lemma 1. *If $x, y \geq 0$ and $p > 1$, then $px^{p-1}(x-y) \geq x^p - y^p$, equality holds if and only if $x = y$. If $b \geq 0$, $a + b \geq 0$ and $p > 1$, then $(a+b)^p \geq pab^{p-1} + b^p$, equality holds if and only if $a = 0$.*

By means of this Lemma, we can show the following necessary condition for the existence of a nonnegative nondecreasing solution of (3).

Theorem 2. *Suppose $y = \{y_k\}_0^{n+1}$ is an admissible solution of (3) such that $y_k > 0$ for $1 \leq k \leq n$. Then for any admissible sequence $u = \{u_k\}_0^{n+1}$, we have $J[u] \geq 0$, equality holds if and only if one of the following conditions holds: (1) $u \equiv 0$, or (2) y is a solution of*

$$\Delta(r_{k-1}(\Delta y_{k-1})^{p-1}) + s_{k-1} y_k^{p-1} = 0, \quad k = 1, 2, \cdots, n, \tag{3$'$}$$

and u is a constant but non-zero multiple of y.

Proof. Let $z_k = u_k/y_k$ for $1 \le k \le n$. Then $z_k \ge 0$ and $u_k = z_k y_k$ for $1 \le k \le n$. Since u is admissible, thus

$$0 \le \Delta u_k = y_k \Delta z_k + z_{k+1}\Delta y_k$$

for $k = 1, 2, \cdots, n-1$. It follows from the admissibility of y and Lemma 1 that

$$\begin{aligned}(\Delta u_k)^p &= (y_k \Delta z_k + z_{k+1}\Delta y_k)^p \\ &\ge p y_k \Delta z_k (z_{k+1}\Delta y_k)^{p-1} + (z_{k+1}\Delta y_k)^p \\ &= y_k\{p z_{k+1}^{p-1}\Delta z_k\}(\Delta y_k)^{p-1} + (z_{k+1}\Delta y_k)^p \\ &\ge y_k(\Delta z_k^p)(\Delta y_k)^{p-1} + (z_{k+1}\Delta y_k)^p\end{aligned}$$

for $k = 1, \cdots, n-1$, where, by Lemma 1, the first inequality is an equality if and only if

$$y_k \Delta z_k = 0, \quad k = 1, \cdots, n-1$$

and the second is an equality if and only if

$$\Delta z_k = 0, \quad k = 1, \cdots, n-1.$$

As a consequence, we have

$$\sum_{k=1}^{n-1} r_k(\Delta u_k)^p \ge \sum_{k=1}^{n-1} r_k y_k (\Delta z_k^p)(\Delta y_k)^{p-1} + \sum_{k=1}^{n-1} r_k (z_{k+1}\Delta y_k)^p \tag{8}$$

and equality holds if and only if $\Delta z_k = 0$ for $k = 1, 2, \cdots, n-1$ (since $y_k > 0$ for $1 \le k \le n-1$). The conditions that $\Delta z_k = 0$ for $k = 1, 2, \cdots, n-1$ is equivalent to $u = cy$. Indeed, if $\Delta z_k = 0$ for $k = 1, \cdots, n-1$, then $u_k = c y_k$ for $1 \le k \le n$. But then $u_0 = -\alpha u_1 = -\alpha c y_1 = c y_0$ and similarly that $u_{n+1} = c y_{n+1}$. The converse is clearly true.

Note that the first sum on the right hand side of (8), if we apply the Abel's transformation, changes to

$$\begin{aligned}
&\sum_{k=1}^{n-1} \Delta(r_k y_k (\Delta y_k)^{p-1} z_k^p] - \sum_{k=1}^{n-1} z_{k+1}^p \Delta[y_k r_k (\Delta y_k)^{p-1}] \\
&= r_n \left[\frac{\Delta y_n}{y_n}\right]^{p-1} u_n^p - r_1 (\Delta y_1)^{p-1} y_1 z_1^p \\
&\quad - \sum_{k=1}^{n-1} z_{k+1}^p y_{k+1} \Delta[r_k (\Delta y_k)^{p-1}] - \sum_{k=1}^{n-1} r_k (z_{k+1} \Delta y_k)^p \\
&= r_n (-1-\beta)^{p-1} u_n^p - \{y_1 z_1^p \Delta[(r_0 (\Delta y_0)^{p-1}] + r_0 (1+\alpha)^{p-1} u_1^p\} \\
&\quad - \sum_{k=1}^{n-1} z_{k+1}^p y_{k+1} \Delta[r_k (\Delta y_k)^{p-1}] - \sum_{k=1}^{n-1} r_k (z_{k+1} \Delta y_k)^p
\end{aligned}$$

where we have used (4) and (5) in obtaining the last equality.

Thus

$$\begin{aligned}
\sum_{k=1}^{n-1} r_k (\Delta u_k)^p \geq - \sum_{k=0}^{n-1} z_{k+1}^p y_{k+1} \Delta[r_k (\Delta y_k)^{p-1}] + r_n (-1-\beta)^{p-1} u_n^p \\
- r_0 (1+\alpha)^{p-1} u_1^p,
\end{aligned}$$

where equality holds if and only if $u_k = c y_k$ for $0 \leq k \leq n+1$. Since by (3),

$$- \sum_{k=0}^{n-1} z_{k+1}^p y_{k+1} \Delta[r_k (\Delta y_k)^{p-1}] \geq \sum_{k=0}^{n-1} z_{k+1}^p y_{k+1} s_k y_{k+1}^{p-1},$$

$J[u] \geq 0$, and equality holds if and only if, $u_k = c y_k$ (i.e. $z_k = c$) for $0 \leq k \leq n+1$ and

$$\sum_{k=0}^{n-1} z_{k+1}^p y_{k+1} \{\Delta[r_k (\Delta y_k)^{p-1}] + s_k y_{k+1}^{p-1}\} = 0,$$

if and only if, either $c = 0$, or, $c \neq 0$ and $(3)'$ holds. The proof is complete.

Similarly, consider the following recurrence relation of the form

$$\Delta(r_{k-1} (\Delta y_{k-1})^{p-1}) + s_{k-1} y_k^{p-1} \leq 0, \quad k = 1, 2, \cdots \tag{9}$$

where $p > 1$, $r_k > 0$ for $k \geq 0$, and $s_k \in R$ for $k \geq 0$. A solution of (9) is a real sequence $\{y_k\}_{k=0}^{\infty}$ which satisfies (9).

Given a fixed real numbers $\alpha \geq -1$, a real sequence $y = \{y_k\}_{k=0}^{\infty}$ is said to be admissible if it satisfies the following conditions:

$$y_0 + \alpha y_1 = 0, \tag{10}$$

$$y_k \geq 0 \quad \text{for} \quad k \geq 1, \tag{11}$$

$$\Delta y_k \geq 0 \quad \text{for} \quad k \geq 1. \tag{12}$$

For any admissible sequence $v = \{v_k\}_{k=0}^{\infty}$, define the functional

$$H[v] = r_0(1+\alpha)^{p-1}v_1^p + \sum_{k=1}^{\infty} r_k(\Delta v_k)^p - \sum_{k=1}^{\infty} s_{k-1}v_k^p.$$

Theorem 3. *Suppose $y = \{y_k\}_0^{\infty}$ is an admissible solution of (9) such that $y_k > 0$ for $k \geq 1$. Then for any admissible sequence $u = \{u_k\}_0^{\infty}$ such that*

$$\sum_{k=1}^{\infty} r_k(\Delta u_k)^p < \infty,$$

we have $H[u] \geq 0$, equality holds if and only if one of the following conditions holds: (1) $u \equiv 0$*; or* (2) y *is a solution of*

$$\Delta(r_{k-1}(\Delta y_{k-1})^{p-1}) + s_{k-1}y_k^{p-1} = 0, \quad k = 1, 2, \cdots \tag{9$'$}$$

and u is a constant but non-zero multiple of y.

The proof is similar to that of Theorem 2. Let $z_k = u_k/y_k$ for $k \geq 1$. Then

$$\sum_{k=1}^{\infty} r_k(\Delta u_k)^p \geq \sum_{k=1}^{\infty} r_k y_k(\Delta z_k^p)(\Delta y_k)^{p-1} + \sum_{k=1}^{\infty} r_k(z_{k+1}\Delta y_k)^p \tag{13}$$

and equality holds if and only if $u_k = cy_k$ for $k \geq 0$. Also,

$$\sum_{k=1}^{\infty} r_k y_k(\Delta z_k^p)(\Delta y_k)^{p-1}$$
$$= -r_0(1+\alpha)^{p-1}u_1^p - \sum_{k=0}^{\infty} z_{k+1}^p y_{k+1}\Delta[r_k(\Delta y_k)^{p-1}] - \sum_{k=1}^{\infty} r_k(z_{k+1}\Delta y_k)^p.$$

Thus

$$\sum_{k=1}^{\infty} r_k(\Delta u_k)^p \geq \sum_{k=1}^{\infty} s_{k-1}u_k^p - r_0(1+\alpha)^{p-1}u_1^p,$$

where equality holds if and only if, $u = cy$ and

$$\sum_{k=0}^{\infty} z_{k+1}^p y_{k+1}\{\Delta[r_k(\Delta y_k)^{p-1}] + s_k y_{k+1}^{p-1}\} = 0,$$

if and only if either $c = 0$, or $c \neq 0$ and (9)' holds. The proof is complete.

We assert that Theorem 1 follows from Theorem 2 and 3. To see this, we first show that the following recurrence relation

$$\Delta(\Delta y_{k-1})^{p-1} + \left[\frac{p-1}{p}\right]^p \left[\frac{1}{k}\right]^p y_k^{p-1} < 0, \quad k = 1, 2, \cdots,$$

has a solution $w = \{w_k\}_0^\infty$ such that $w_0 = 0$, $w_1 = 1$, $w_k > 0$ and $\Delta w_k > 0$ for $k \geq 1$, and $\Delta w_k / w_k \to 0$ as $k \to \infty$.

Indeed, let $w_0 = 0$, $w_k = k^{(p-1)/p}$ for $k \geq 1$. Then by the mean value theorem,

$$\Delta w_k = (k+1)^{(p-1)/p - k(p-1)/p} = \left[\frac{p-1}{p}\right] \mu_k^{-1/p}, \quad k < \mu_k < k+1$$

so that

$$\Delta w_k < \left[\frac{p-1}{p}\right] k^{-1/p}.$$

Similarly,

$$\Delta w_{k-1} = \left[\frac{p-1}{p}\right] \mu_{k-1}^{-1/p}, \quad k-1 < \mu_{k-1} < k.$$

Thus

$$\Delta(\Delta w_{k-1})^{p-1} < \left[\frac{p-1}{p}\right]^{p-1} \left[k^{-(p-1)/p} - \mu_{k-1}^{-(p-1)/p}\right]$$
$$= -\left[\frac{p-1}{p}\right]^p \mu^{(1-2p)/p}, \quad k-1 < \mu < k.$$

It follows that

$$\Delta(\Delta w_{k-1})^{p-1} + \left[\frac{p-1}{p}\right]^p \left[\frac{1}{k}\right]^p w_k^{p-1}$$
$$< -\left[\frac{p-1}{p}\right]^p \mu^{(1-2p)/p} + \left[\frac{p-1}{p}\right]^p \left[\frac{1}{k}\right]^p k^{(p-1)^2/p}$$
$$= \left[\frac{p-1}{p}\right]^p \left[k^{(1-2p)/p} - \mu^{(1-2p)/p}\right] < 0$$

as desired. The fact that $w_{k+1}/w_k \to 1$ is clear. Our assertion is proved.

Now let $a_k \geq 0$ for $k = 1, 2, \cdots, n$. Let $A_0 = 0$, $A_1 = a_1$, $A_2 = a_1 + a_2, \ldots$, $A_n = a_1 + \ldots + a_n$ and $A_{n+1} = -\beta_n A_n$ where $\beta_n = -w_{n+1}/w_n < -1$. Then $\{A_k\}_0^{n+1}$ is an admissible sequence with respect to $\alpha = 0$ and $\beta = \beta_n$. Thus by Theorem 2,

$$\sum_{k=1}^{n} a_k^p \geq \sum_{k=1}^{n} \left[\frac{p-1}{p}\right]^p \left[\frac{A_k}{k}\right]^p + \left[\frac{w_{n+1}}{w_n} 1\right]^{p-1} A_n^p$$

where equality holds if and only if $A_k = 0$ for $0 \leq k \leq n+1$. This implies

$$\sum_{k=1}^{n} a_k^p > \sum_{k=1}^{n} \left[\frac{p-1}{p}\right]^p \left[\frac{A_k}{k}\right]^p$$

unless $a_k = 0$ for $1 \leq k \leq n$, which extends (1).

Similarly, let $a_k \geq 0$ for $k \geq 1$, $A_0 = 0$, $A_k = a_1 + \ldots + a_k$ for $k \geq 1$. Then $\{A_k\}_0^\infty$ is an admissible sequence with respect to $\alpha = 0$. If

$$\sum_{n=1}^{\infty} a_n^p < \infty,$$

then by Theorem 3,

$$\sum_{k=1}^{\infty} a_k^p \geq \sum_{k=1}^{\infty} \left[\frac{p-1}{p}\right]^p \left[\frac{A_k}{k}\right]^p,$$

where equality holds if and only if $A_k = 0$ for $k \geq 0$. Thus (2) holds unless $a_k = 0$ for $k \geq 1$.

Acknowledgement

The first author is supported by NSC of Republic of China.

References

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