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THE SPECTRUM OF THE LAPLACE OPERATOR FOR A GENERALIZED DOLD MANIFOLD

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1. Introduction:

Let (M,g), (N,h) be two compact and orientable Riemannian manifolds. Let Γ be a finite subgroup of the group of isometries $I(M\times N)$ which acts freely on the manifold $M\times N$. One of the problems of the spectrum is to determine the $Sp(M\times N/\Gamma)$.

The aim of the present paper is to determine the $Sp(S^n\times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2)$, where $S^n\times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2$ is called Dold manifold.

The whole paper contains four paragraphs.

The second paragraph deals firstly with the spectrum of the manifold $(M\times N,\, g\times h)$. It also gives the general theory about $Sp(M\times N/\Gamma)$, where Γ is a subgroup of $I(M\times N)$ which is isomorphic onto $\mathbf{Z}_2$.

The spectrum of the Dold manifold $S^n\times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2$ is computed in the third paragraph.

The last paragraph deals with the conditions such that $Sp(S^n\times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2)$ determines the geometry on $S^n\times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2$.

§ **2.** We consider two compact orientable Riemannian manifolds (M,g) and (N,h). From these manifolds we obtain the manifold $(M\times N,\, g\times h)$. We have the following diagram:

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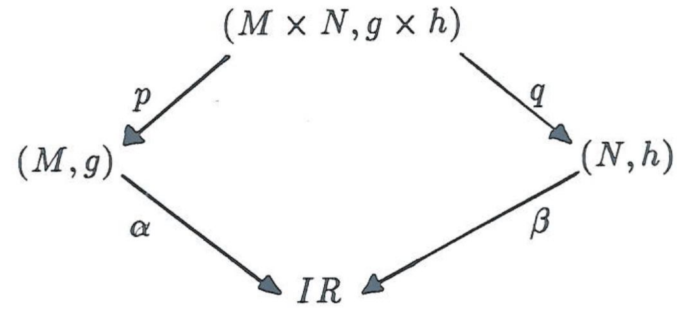

If α and β are two functions on M and N respectively, then αop and βoq are functions on $M \times N$. We denote by $\Delta^{M\times N}$, Δ^M and Δ^N the Laplace operators on the Riemannian manifolds $(M \times N,\, g \times h)$, (M,g) and (N,h) respectively. It can easily be proved

$$\Delta^{M\times N}[(\alpha\, op) \times (\beta\, oq)] = (\beta\, oq) \times [\Delta^M(\alpha)\, op] + (\alpha\, op) \times [\Delta^N(\beta) oq] \quad (2.1)$$

If α is an eigenfuction of Δ^M with eigenvalue λ and β is an eigenfuction of Δ^N with eigenvalue μ, then the relation (2.1) takes the form

$$\Delta^{M\times N}[(\alpha\, op) \times (\beta\, oq)] = (\lambda + \mu)[(\alpha\, op) \times (\beta\, oq)]$$

that means $(\alpha\, op) \times (\beta\, oq)$ is an eigenfunction for the Laplace operator $\Delta^{M\times N}$ with eigenvalue $\lambda + \mu$.

Let (M', g') be a compact and orientable Riemannian manifold with metric g'. We denote by $C^\infty(M')$ the $\mathbf{C}$-algebra of all differentiable functions on M'. From M' we obtain the $Sp(M',g') = \{\lambda / \Delta^{M'} f = \lambda f,\ f \in C^\infty(M'),\ \lambda \in \mathbf{R}\}$. We consider the subspace $Q(M',g')$ of $C^\infty(M')$ defined by

$$Q(M',g') = \sum_{\lambda \in Sp(M',g)} Q_\lambda(M',g')$$

where $Q_\lambda(M',g')$ is the vector subspace of $C^\infty(M')$ consisting of the eigefunctions with eigenvalue λ.

$Q_\lambda(M',g')$ is called eigen-subspace of (M',g') and has finite dimension. The subset $Q(M',g')$ is dense in $C^\infty(M')$ provided with the topology defined by the inner product $< f_1, f_2 >= \int_{M'} f_1 f_2 dM'$, where dM' is the volume element on M'.

We consider the subspace of $C^\infty(M \times N)$ generated by all products of the form $(\alpha\, op) \times (\beta\, oq)$, where $\alpha \in Q\,(M, g)$ and $\beta \in Q(N, h)$. This vector subspace of $C^\infty(M \times N)$ is denoted by $p^\star Q(M, g) \otimes g^\star Q(N, h)$. This is isomorphic onto the vector space $Q(M, g) \otimes Q(N, h)$. The following relations are true.

(i) $$p^\star Q(M,g) \otimes q^\star Q(N,h) = Q(M \times N,\, g \times h)$$

(ii) $$Sp(M \times N,\, g \times h) = \{\lambda + \mu / \lambda \in Sp(M,g), \mu \in Sp(N,h)\}$$

(iii) $$Q_\nu(M \times N,\, g \times h) = \sum_{\substack{\lambda \in Sp(M,g)\\ \mu \in Sp(N,h)\\ \lambda+\mu=\nu}} p^\star Q_\lambda(M,g) \otimes q^\star Q_\mu(N,h)$$

Theorem 2.1. *Let (M, g), (n, h) be two compact and orientable Riemannian manifolds. Let $\Gamma \simeq \mathbf{Z}_2$ be the group of isometries on the manifold $(M \times N,\, g \times h)$ which acts freely on $M \times N$ such that preserves the seperate coordinate system on the product manifold $M \times N$. Then the eigenfunctions on the manifold $(M \times N/\Gamma,\, g \times h/\Gamma)$ are of the form $(\alpha\, op) \times (\beta\, oq)$ where α and β are the eigenfunctions on the manifolds (M, g) and (N, h) respectively which are invariant under the action by the group Γ or takes opposite sign under the action of Γ.*

Proof. Let $(U_i, \varphi_i)_{i \in I}$ and $(V_j, \psi_j)_{j \in J}$ be two atlas on (M, g) and (N, h) respectively. Then $(U_i \times U_j, \varphi_i \times \psi_j)_{(i,j) \in I X J}$ is an atlas on the Riemannian manifold $(M \times N,\, g \times h)$. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ be two local coordinate systems on the charts (U_i, φ_i) and (V_j, ψ_j) respectively, where dim $M = n$ and dim $N = m$. Then $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a local coordinate system on the chart $(U_i \times V_j,\, \varphi_i \times \psi_j)$, If γ is an element of the group Γ, then we have

$$\gamma : M \times N \to M \times N, \quad \gamma :\ U_i \times V_j \to U_i' \times V_j'$$

$$\begin{aligned}\gamma : (x_1, \ldots, x_n; y_1, \ldots, y_m) &\to (\gamma(x_1, \ldots, x_n); \gamma(y_1, \ldots, y_m))\\ &= (x_1', \ldots, x_n'; y_1', \ldots, y_m')\end{aligned}$$

which means that γ preserves the separate coordinate systems. We assume that

$$\alpha, \alpha_1 \in Q_\lambda(M,q) \quad \text{and} \quad \beta, \beta_1 \in Q_\mu(N,h)$$

then we have

$$\begin{aligned}
\Delta^{M\times N}[(\alpha\, op)\times(\beta\, oq)] &= (\lambda+\mu)[(\alpha\, op)\times(\beta\, oq)] \\
\Delta^{M\times N}[(\alpha_1\, op)\times(\beta_1\, oq)] &= (\lambda+\mu)[(\alpha_1\, op)\times(\beta_1\, oq)]
\end{aligned}$$

If the functions α and β are invariant under the action of the group Γ and α_1 and β_1 take opposite sing by the action of the group Γ, then we have

$$\begin{aligned}
\gamma(\alpha(x_1,\dots,x_n)) &= \alpha(\gamma(x_1,\dots,x_n)) = \alpha(x_1',\dots,x_n') = \alpha(x_1,\dots,x_n) \\
\gamma(\alpha_1(x_1,\dots,x_n)) &= \alpha_1(\gamma(x_1,\dots,x_n)) = \alpha_1(x_1',\dots,x_n') = -\alpha_1(x_1,\dots,x_n) \\
\gamma(\beta(y_1,\dots,y_m)) &= \beta(\gamma(y_1,\dots,y_m)) = \beta(y_1',\dots,y_m') = \beta(y_1,\dots,y_m) \\
\gamma(\beta_1(y_1,\dots,y_m)) &= \beta_1(\gamma(y_1,\dots,y_m)) = \beta(y_1',\dots,y_m') = -\beta_1(y_1,\dots,y_m)
\end{aligned}$$

then we have

$$\begin{aligned}
&\Delta^{M\times N/\Gamma}\left[(\alpha\, op)\times(\beta\, oq)\right] = \Delta^{M\times N}\left[(\gamma(\alpha)op)\times(\gamma(\beta)oq)\right] \\
&= \Delta^{M\times N}\left[(\alpha\, op)\times(\beta\, oq)\right] = (\lambda+\mu)\left[(\alpha\, op)\times(\beta\, oq)\right] \\
&\Delta^{M\times N/\Gamma}\left[(\alpha_1\, op)\times(\beta_1\, oq)\right] = \Delta^{M\times N}\left[(\gamma(\alpha_1)op)\times(\gamma(\beta_1)oq)\right] \\
&= \Delta^{M\times N}\left[|\,(-\alpha_1\, op)\times(-\beta_1\, oq)\right] = (\lambda+\mu)\left[(\alpha_1\, op)\times(\beta_1\, oq)\right]
\end{aligned}$$

3. It is known that the Dold manifold is defined by

$$M' = S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2$$

where the action of the group $\mathbf{Z}_2$ is given as follows

$$\{(x_1,\dots,x_n),(z_0,z_1,\dots,z_m)\} \to \{(-x_1,\dots,-x_n),(\overline{z}_0,\overline{z}_1,\dots,\overline{z}_m)\} \tag{3.1}$$

The manifold M' can be obtained by identification of the two points defined by (3.1) on the manifold $S^n \times \mathbf{P}^m(\mathbf{C})$.

Theorem 3.1. *Let $M' = S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2$ be the Dold manifold provided with the metric $g_0 \times h_0$, where g_0 is the standard metric on S^n with constant sectional curvature 1 and h_0 is the Study-Fubini metric on $\mathbf{P}^m(\mathbf{C})$ with constant holomrphic sectional curvature 1. Then, the Spectrum of this manifold has the form $Sp(S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2, g_0 \times h_0/\mathbf{Z}_2) = \{0, \lambda_{2k-1} + \mu_{2\rho-1}, \lambda_{2k} + \mu_{2\rho-1}, \lambda_{2k} + \mu_{2\rho}$, where $\lambda_\nu = \nu(n+\nu-1)$, $\mu_\sigma = \sigma(m+\sigma)$ with multipilicities $q_\nu(\lambda_\nu) = \frac{(n+\nu-2)...(n+1)n(n+2\nu-1)}{\nu!}$, $q_\sigma(\mu_\sigma) = A(-m,\sigma)$ if σ is odd and $q_\sigma(\mu_\sigma) = 2A(m,\sigma)$, if σ is even, where $A(m,\sigma) = \frac{2(2m+\sigma-1)(2m+\sigma)...(2m+2)(2m+1)(m+\sigma)}{\sigma!}$, respectively.*

Proof. If is known the eigenfuctions of Δ^{S^n} can be obtained by the restriction on S^n the harmonic polynomials on $\mathbf{R}^{n+1}$. The harmonic polynomials which are invariant under the action of the group $\mathbf{Z}_2$ are the even dimensional. The harmonic polynomials which take opposite sign by the action of the group $\mathbf{Z}_2$ are the odd dimension. Finally, we conclude that all the harmonic polynomials of $\mathbf{R}^{n+1}$ give eigenvalues for $\Delta^{S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2}$.

Therefore we obtain

$$\lambda_k = k(n+k-1),\ q_k(\lambda_k) : (n+k-2)\cdots(n+1)n(n+2k-1)/k!$$

The complex projective space $\mathbf{P}^m(\mathbf{C})$ is the base space of the following fibre

$$S^1 \xrightarrow{i} S^{2m+1} \xrightarrow{\pi} \mathbf{P}^m(\mathbf{C}) \tag{3.4}$$

Therefore the eigenfunctions of $\Delta^{\mathbf{P}^m(\mathbf{C})}$ are the eigenfunctions of S^{2m+1} which are invariants by the action of S^1, or the harmoric polynomials in $\mathbf{R}^{2m+2} \simeq \mathbf{C}^{m+1}$ which are invariant by the action of S^1. At the same time we try to determine which of these polynomials are invariants, by the action of the conjugation and the others taking opposite sign by conjugation.

If $(z_0, z_1, \ldots, z_m)$ and $(x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_m)$ are the natural coordinate systems on $\mathbf{C}^{m+1}$ and $\mathbf{R}^{2m+2}$ respectively, then we have the relations

$$z_0 = x_0 + \sqrt{-1}y_0,\ z_1 = x_1 + \sqrt{-1}y_1,\ \ldots,\ z_m = x_m + \sqrt{-1}y_m \tag{3.5}$$

From (3.5) we obtain

$$\overline{z}_0 = x_0 - \sqrt{-1}y_0,\ \overline{z}_1 = x_1 - \sqrt{-1}y_1,\ \ldots\ \overline{z}_m = x_m - \sqrt{-1}y_m \tag{3.6}$$

The relations (3.5) and (3.6) imply

$$x_0 = \frac{1}{2}(z_0 + \overline{z}_0),\ x_1 = \frac{1}{2}(z_1 + \overline{z}_1), \ldots, x_m = \frac{1}{2}(z_m + \overline{z}_m) \tag{3.7}$$

$$y_0 = \frac{1}{2\sqrt{-1}}(z_0 - i\overline{z}_0),\ y_1 = \frac{1}{2\sqrt{-1}}(z_1 - i\overline{z}_1), \ldots, y_m = \frac{1}{2\sqrt{-1}}(z_m - i\overline{z}_m) \tag{3.8}$$

Therefore $(z_0, z_1, \ldots, z_m, \overline{z}_0, \overline{z}_1, \ldots, \overline{z}_m)$ can be obtained as a coordinate system on $\mathbf{R}^{2m+2}$. The harmonic polynomials on $\mathbf{R}^{2m+2} \simeq \mathbf{C}^{m+1}$ are polynomials in $z = (z_1, z_2, \ldots, z_m)$ and in $\overline{z} = (\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_m)$. The Laplace operator on $\mathbf{R}^{2m+2} = \mathbf{C}^{m+1}$ has the form

$$\triangle = -\frac{1}{4}\sum_{p=1}^{m+1} \frac{\partial}{\partial z_p}\frac{\partial}{\partial \overline{z}_p} \tag{3.9}$$

The polynomial $Q = Q(z, \overline{z}) = Q(z_0, z_1, \ldots, z_m, \overline{z}_0, \overline{z}_1, \ldots, \overline{z}_m)$ is harmonic if we have

$$\sum_{p=1}^{m+1} \frac{\partial^2}{\partial z_p}\frac{Q}{\partial \overline{z}_p} = 0 \tag{3.10}$$

The harmonic polynomials $Q(z, \overline{z})$, which are invariant by the action of S^1 are the same degree with respect to z and $\overline{z}$. All these polynomials are denoted by $H_{p,p}$. Therefore the eigenvalue which corresponds to each of the eigenfunctions of $\triangle^{S^{2m+1}}$ which are at the same time eigenfunctions of $\triangle^{\mathrm{P}^m(\mathbf{C})}$ has the form

$$\mu_p = 2p(2m + 1 + 2p - 1) = 4p(m + p), \quad p \geq 0.$$

From these polynomials we must take those remaining invariant by conjugation and the others taking opposite sign by conjugation. The first polynomials form a vector space of the same dimension as the vector space H_p, when p is odd. The other polynomials, when p is even, form a vector space whose dimension is $2\dim H_p$. Therefore, we have the following eigenvalues for the Dold manifold

$\Delta^{S^n} \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2$:

$0, \lambda_{2k-1} + \mu_{2p-1}, \lambda_{2k} + \mu_{2p-1}, \lambda_{2k} + \mu_{2p}$, where $\lambda_\nu = \nu(\nu + n - 1)$, $\mu_\sigma = 4\sigma(\mu + \sigma)$ with multiplicities $q_k(\lambda_k) = (n+k-2)\ldots(n+1)n(n+2k-1)/k!$, $q_\sigma(\mu_\sigma) = 2A(m,\sigma)$ if σ is even and $q_\sigma(\mu_\sigma) = A(m,\sigma)$ if σ is odd, where $A(m,\sigma) = \dfrac{2(2m+p-1)\ldots(2m+1)(m+p)}{p!}$.

4. Let $(M \times N,\, g_1 \times g_2)$ be a compact and orientable Riemannian manifold with the following properties

(i) N can carry a complex structure,

(ii) $\pi_1(M \times N) = \mathbf{Z}_2$

(iii) $Sp(M \times N,\, g_1 \times g_2) = Sp(S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2,\, g_0 \times h_0/\mathbf{Z}_2)$

(vi) g = Kähler metric

We give the conditions under which the Riemannian manifolds $(M \times N,\, g_1 \times g_2)$ and $(S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2,\, g_0 \times h_0/\mathbf{Z}_2)$ are isometric.

It is known that the universal covering of the manifold $S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2$ is $S^n \times \mathbf{P}^m(\mathbf{C})$. We denote by $M' \times N'$ the universal covering of $M \times N$.

These manifolds $M' \times N'$ and $S^n \times \mathbf{P}^m(\mathbf{C})$ have metrics $g_1' \times g_2'$ and $g_0 \times h_0$ respectively such that $g_1' \times g_2'/\mathbf{Z}_2 = g_1 \times g_2$. The manifolds $(M \times N,\, g_1 \times g_2)$ and $(S^n \times \mathbf{P}^M(\mathbf{C})/\mathbf{Z}_2,\, g_0 \times h_0/\mathbf{Z}_2)$ are locally isometric onto the manifolds $(M' \times N',\, g_1' \times g_2')$ and $(S^n \times \mathbf{P}^m(\mathbf{C}),\, g_0 \times h_0)$ respectively. It can easily obtained from $Sp(M \times N,\, g_1 \times g_2)$ and $Sp(S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2,\, g_0 \times h_0/\mathbf{Z}_2)$ that $Sp(S^n \times \mathbf{P}^m(\mathbf{C}),\, g_0 \times h_0)$ and $Sp(M' \times N',\, g_1' \times g_2')$ have the same eigeuvalues. We assume that these eigenvalues have the same multiplicity and therefore we obtain

$$Sp(M' \times N',\, g_1' \times g_2') = Sp(S^n \times \mathbf{P}^m(\mathbf{C}),\, g_0 \times h_0) \qquad (4.2)$$

From (4.2) we obtain

$$Sp(M', g_1') = Sp(S^n, g_0) \qquad (4.3)$$

$$Sp(N', g_2') = Sp(\mathbf{P}^m(\mathbf{C}), h_0) \qquad (4.4)$$

The relations (4.3) and (4.4) imply (M', g_1') is isometric onto (S^n, g_0), if $1 \le n \le 6$ ([2]) and (N', g_2') is beholomorphically isometric onto $(\mathbf{P}^m(\mathbf{C}), h_0)$, if $1 \le m \le 5$, ([4]).

Now we can state the following theorem .

Theorem 4.1. *Let $(M \times N, g_1 \times g_2)$ be a compact, and orientable Riemannian manifold with the properties $((i) - (iv)$ (4.1). Let $(M' \times N', g_1' \times g_2')$ be the universal covering of $M \times N$ such that $g_1 \times g_2 = g_1' \times g_2'/\mathbf{Z}_2$. If $Sp(M \times N, g_1 \times g_2) = Sp(S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2, g_0 \times h_0/\mathbf{Z}_2)$ and the mulitiplicity of the eigenvalues of $Sp(M' \times N', g_1 \times g_2)$ are the same as of $Sp(S^n \times \mathbf{P}^m(\mathbf{C}), g_0 \times h_0)$ and $1 \leq n \leq 6$, $1 \leq m \leq 5$, Then $(M \times N, g_1 \times g_2)$ is isometric onto $(S^n \times \mathbf{R}^m(\mathbf{C})/\mathbf{Z}_2, g_0 \times h_0/\mathbf{Z}_2)$.*

If the conditions (4.1) are substituded by the following

(i) N is beholomorphically isomorphic onto $\mathbf{P}^m(\mathbf{C})$

(ii) $Sp(M \times N, g_1 \times g_2) = Sp(S^n \times \mathbf{P}^m(\mathbf{C}), g_0 \times h_0/\mathbf{Z}_2)$ (4.5)

(iii) Multiplity of the eigenvalues of $Sp(M' \times N', g_1 \times g_2)$ are the same as of $Sp(S^n \times \mathbf{P}^m(\mathbf{C}), g_0 \times g_0)$ then (N', g') coincides with $(\mathbf{P}^m(\mathbf{C}), h_0)$.

From the theorem (4.1) and the conditions (4.5) we conclude the following corollary.

Corollary 4.2. *Let $(M \times N, g_1 \times g_2)$ be a compact and orientable Riemannian manifold with the propporties (4.5), If $Sp(M \times N, g_1 \times g_2) = Sp(S^n \times \mathbf{P}^m(\mathbf{C})/\mathbf{Z}_2, g_0 \times h_0/\mathbf{Z}_2)$, where $2 \leq n \leq 6$, and $1 \leq m \leq 5$, then $(M \times N, g_1 \times g_2)$ is isometric onto $(S^n \times \mathbf{P}^m(\mathbf{C}), g_0 \times h_0)$.*

§ **5.** The manifold $S^{2n-1} \times \mathbf{P}^m(\mathbf{C})/\Gamma$ is called generalized Dold manifold, where *Gamma* is a finite subgroup of order greater or equal than three of the group of isommetries $I(S^{2n-1} \times \mathbf{P}^m(\mathbf{C}))$ of the manifold $S^{2n-1} \times \mathbf{P}^m(\mathbf{C})$.

If we use the same technique as for the manifold $S^n \times \mathbf{P}^m(\mathbf{C})/Z_2$ then we can compute the $S_p(S^{2n-1} \times \mathbf{P}^m(\mathbf{C})/\Gamma)$.

Now we can prove the theorem:

Theorem 5.1. *Let $S^{2n-1} \times \mathbf{P}^m(\mathbf{C})/\Gamma$ be a generalized Dold manifold. If n, m and order of Γ have special prices, then $S_p(S^{2n-1} \times \mathbf{P}^m(\mathbf{C})/\Gamma)$ determines the geometry on the manifold $S^{2n-1} \times \mathbf{P}^m(\mathbf{C}))$.*

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