TOTALLY REAL SURFACES IN $S^6$

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Abstract. The normal bundle $\nu$ of a totally real surface $M$ in $S^6$ splits as $\nu = JTM \oplus \mu$, where $TM$ is the tangent bundle of $M$ and $\mu$ is sub-bundle of $\nu$ which is invariant under the almost complex structure $J$. We study the totally real surfaces $M$ of constant Gaussian curvature $K$ for which the second fundamental form $h(x,y) \in JTM$, and we show that $K = 1$ (that is, $M$ is totally geodesic).

1. Among the Euclidean spheres it is known that $S^2$ and $S^6$ have almost complex structure of which the almost complex structure on $S^2$ is integrable where as that on $S^6$ is not integrable. The almost complex structure on $S^6$ is nearly Kaehler, that is, it satisfies $(\nabla_X J)(X) = 0$, where $\nabla$ is the Riemannian connection on $S^6$ and $J$ is the almost complex structure of $S^6$ (cf.[1]). It is known that $S^6$ has no 4-dimensional complex submanifolds (cf.[2]). The 3-dimensional totally real submanifolds of $S^6$ have been studied by Ejiri (cf.[1]). In the present paper we study the 2-dimensional totally real submanifolds of $S^6$.

Let $M$ be 2-dimensional totally real submanifold of $S^6$ with tangent bundle $TM$ and the normal bundle $\nu$. Since $M$ is totally real we, have for each $X \in TM$, $JX \in \nu$. The normal bundle splits as $\nu = JTM \oplus \mu$, where $\mu$ is sub-bundle of $\nu$, which is invariant under $J$, that is, $J\mu = \mu$. We denote by $g$ the Riemannian metric on $S^6$ as well as that induced on $M$. The Riemannian connection $\nabla$ of

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$S^6$ induces the Riemannian connection $\nabla$ on $M$ as well as the connection $\nabla^\perp$ in the normal bundle $\nu$, and they are related by the following formulae

$$\nabla_X Y = \nabla_X Y + h(X,Y), \quad \nabla_X N = -A_N X + \nabla_X^\perp N, \quad X,Y \in \mathcal{X}(M), \quad N \in \nu,$$

(1.1)

where $h(X,Y)$ and $A_N$ are the second fundamental forms satisfying $g(h(X,Y),N) = g(A_N X,Y)$ and $\mathcal{X}(M)$ is the Lie algebra of the vector fields on $M$.

The Gaussian curvature $K$ of $M$ is given by

$$K = 1 + g(h(X,X),h(Y,Y)) - g(h(X,Y),h(X,Y)),$$

(1.2)

where $\{X,Y\}$ is a local orthonormal frame on $M$. The Codazzi equation gives (cf. [3])

$$(\nabla_X h)(Y,Z) = \nabla_Y h(X,Z), \quad X,Y,Z \in \mathcal{X}(M),$$

(1.3)

where

$$(\nabla_X h)(Y,Z) = \nabla_X^\perp h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

Lemma 1.1. Let $M$ be totally real surface of $S^6$. Then for any local orthonormal frame $\{X,Y\}$, $(\nabla_X J)(Y)$ is unit normal vector field in $\nu$.

**Proof.** Since $M$ is totally real, it follows that $\{JX,JY\}$ is orthonormal in $\nu$. Also $\nabla_X J$ is skew-symmetric and $(\nabla_X J)(X) = 0$. Then we easily get that $g((\nabla_X J)(Y),X) = 0$, $g((\nabla_X J)(Y),Y) = 0$, $g((\nabla_X J)(Y),JX) = 0$, $g((\nabla_X J)(Y),JY) = 0$, where we have used $(\nabla_X J)(Y) = -(\nabla_Y J)(X)$ and $(\nabla_X J)(JY) = -J(\nabla_X J)(Y)$, which is consequence of $(\nabla_X J)(X) = 0$. This proves that $(\nabla_X J)(Y) \in \nu$. $(\nabla_X J)(Y)$ is a unit vector follows directly from Lemma (cf. [1], p. 760).

2. We shall be concerned with 2-dimensional totally real submanifold $M$ of $S^6$ for which $h(X,Y) \in JTM$, $X,Y \in \mathcal{X}(M)$. With the help of (1.1), we obtain

$$(\nabla_X J)(Y) = \nabla_X^\perp JY - A_J Y X - J \nabla_X Y - J h(X,Y).$$

Since $(\nabla_X J)(Y) \in \nu$, and by our assumption $Jh(X,Y)$ is tangent to $M$, on equating tangential and normal components in above equation we obtain

$$\nabla_X^\perp JY = J \nabla_X Y + (\nabla_X J)(Y), \quad h(X,Y) = JA_J Y X, \quad X,Y \in \mathcal{X}(M).$$

(2.1)
From the second equation in (2.1) using the symmetry of \( h(X,Y) \), we get \( A_{JY}X = A_{JX}Y \) and consequently we obtain

\[
g(h(X,Y), JZ) = g(h(Y,Z), JX) = g(h(X,Z), JY). \tag{2.2}
\]

**Theorem 2.1.** Let \( M \) be a 2-dimensional totally real submanifold of constant Gaussian curvature of \( S^6 \) with \( h(X,Y) \in JTM \). Then the Gaussian curvature of \( M \) is 1, that is, \( M \) is totally geodesic.

**Proof.** Let \( UM = \{X \in TM : \|X\| = 1\} \) be the unit tangent bundle of \( M \). Consider the function \( f : UM \to \mathbb{R} \) defined by \( f(X) = g(h(X,X), JX) \), which is clearly smooth. We claim that if \( f \) is constant, then \( M \) has Gaussian curvature 1. Since \( f(-X) = -f(X) \), if \( f \) is constant, then \( f = 0 \), and consequently \( g(h(X,X), JX) = 0 \), \( X \in TM \). For any orthonormal frame \( \{X, Y\} \) of \( M \), we shall get

\[
g(h(X,X), JX) = 0, \quad g(h(Y,Y), JY) = 0, \quad g(h(\frac{X+Y}{\sqrt{2}}, \frac{X+Y}{\sqrt{2}}), J(\frac{X+Y}{\sqrt{2}})) = 0, \quad g(h(\frac{X-Y}{\sqrt{2}}, \frac{X-Y}{\sqrt{2}}), J(\frac{X-Y}{\sqrt{2}})) = 0.
\]

Using (2.2) in these equations we get \( g(h(X,Y), JX) = 0 \) and \( g(h(X,Y), JY) = 0 \). Since \( h(X,Y) \in JTM \) and \( \{JX, JY\} \) is an orthonormal frame of \( JTM \), we get \( h(X,X) = 0, h(Y,Y) = 0, h(X,Y) = 0 \), that is, we get that \( M \) is totally geodesic and hence by (1.2), the Gaussian curvature of \( M \) is 1

Suppose \( f \) is not a constant. The unit tangent bundle \( UM \) being compact, \( f \) attains maximum. Suppose \( f \) attains maximum at \( e_1 \). Then it is known that \( g(h(e_1,e_1), JY) = 0 \) for \( Y \perp e_1, Y \in UM \) (cf.[1]). Put \( h(e_1,e_1) = \lambda Je_1 \), where \( \lambda \) is a smooth function on \( M \). Choose \( e_2 \) such that \( \{e_1, e_2\} \) is a local orthonormal frame of \( M \). Then using (2.2) we have the following expressions

\[
h(e_1,e_1) = \lambda Je_1, \quad h(e_2,e_2) = \mu Je_1 + \nu Je_2, \quad h(e_1,e_2) = \mu Je_2, \tag{2.3}
\]

where \( \mu, \nu \) are smooth functions on \( M \).

From the structure equations of \( M \) we have the following local equations

\[
\nabla_{e_1} e_1 = a e_2, \quad \nabla_{e_2} e_2 = b e_1, \quad \nabla_{e_1} e_2 = -ae_1, \quad \nabla_{e_2} e_1 = -be_2, \tag{2.4}
\]

where $a, b$ are smooth functions.

Taking (1.3) in form of the following two equations

\[(\nabla_{e_1} h)(e_1, e_2) = (\nabla_{e_2} h)(e_1, e_1), \quad (\nabla_{e_2} h)(e_1, e_2) = (\nabla_{e_1} h)(e_2, e_2)\]

and using (2.1), (2.3) and (2.4), on equating the components we obtain

\[
e_1 \cdot \mu = a \nu - b(\lambda - 2\mu), \quad e_2 \cdot \lambda = a(\lambda - 2\mu), \quad (\lambda - \mu)(\nabla_{e_1} J)(e_2) = 0 \quad (2.5)
\]

\[
e_2 \cdot \mu - e_1 \cdot \nu = -b \nu + 3a \mu, \quad \nu(\nabla_{e_1} J)(e_2) = 0. \quad (2.6)
\]

Since from Lemma 1.1, $(\nabla_{e_1} J)(e_2)$ is unit vector, we get that $\lambda = \mu$ and $\nu = 0$. The Gaussian curvature $K$ of $M$ is then found from (1.2) and (2.3) as $K = 1 + \lambda^2 - \lambda^2 = 1.$ This proves the Theorem.

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References


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