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TOTALLY REAL SURCACES IN S^6

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Abstract. The normal bundle $\overline{\nu}$ of a totally real surface M in S^6 splits as $\overline{\nu} = \text{JTM} \oplus \overline{\mu}$, where TM is the tangent bundle of M and $\overline{\mu}$ is sub-bundle of $\overline{\nu}$ which is invariant under the almost complex structure J. We study the totally real surfaces M of constant Gaussian curvature K for which the second fundamental form $h(x, y) \in \text{JTM}$, and we show that $K = 1$ (that is, M is totally geodesic).

1. Among the Euclidean spheres it is known that S^2 and S^6 have almost complex structure of which the almost complex structure on S^2 is integrable where as that on S^6 is not integrable. The almost complex sctructure on S^6 is nearly Kaehler, that is, it satisfies $(\overline{\nabla}_X J)(X) = 0$, where $\overline{\nabla}$ is the Riemannian connection on S^6 and J is the almost complex structue of S^6 (cf.[1]). It is known that S^6 has no 4-dimensional complex submanifolds (cf.[2]). The 3-dimensional totally real submanifolds of S^6 have been studied by Ejiri (cf.[1]). In the present paper we study the 2-dimensional totally real submanfiolds of S^6.

Let M be 2-dimensional totally real submanifold of S^6 with tangent bundle TM and the normal bundle $\overline{\nu}$. Since M is totally real we, have for each $X \in$ TM, $JX \in \overline{\nu}$. The normal bundle splits as $\overline{\nu} = J\text{TM} \oplus \overline{\mu}$, where $\overline{\mu}$ is sub-bundle of $\overline{\nu}$, which is invariant under J, that is, $J\overline{\mu} = \overline{\mu}$. We denote by g the Riemannian metric on S^6 as well as that induced on M. The Riemannian connection $\overline{\nabla}$ of

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S^6 induces the Riemannian connection ∇ on M as well as the connection $\nabla^\perp$ in. the normal bundle $\overline{\nu}$, and they are related by the following farmulae

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \overline{\nabla}_X N = -A_N X + \nabla^\perp_X N,\ X,Y \in \mathcal{X}(M),\ N \in \overline{\nu}, \tag{1.1}$$

where $h(X,Y)$ and A_N are the second fundamental forms satisfying $g(h(X,Y),N) = g(A_N X, Y)$ and $\mathcal{X}(M)$ is the Liealgebra of the vector fields on M.

The Gaussian curvature K of M is given by

$$K = 1 + g(h(X,X),h(Y,Y)) - g(h(X,Y),h(X,Y)), \tag{1.2}$$

where $\{X,Y\}$ is a local orthonormal frame on M. The Codazzi equation gives (cf.[3])

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) \quad X,Y,Z \in \mathcal{X}(M), \tag{1.3}$$

where $\quad (\overline{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

Lemma 1.1. *Let M be totally real surface of S^6. Then for any local orthonormal frame $\{X,Y\}$, $(\overline{\nabla}_X J)(Y)$ is unit normal vector field in $\overline{\mu}$.*

Proof. Since M is totally real, it follows that $\{JX, JY\}$ is orthonormal in $\overline{\nu}$. Also $\overline{\nabla}_X J$ is skewsymmetric and $(\overline{\nabla}_X J)(X) = 0$. Then we easily get that $g((\overline{\nabla}_X J)(Y), X) = 0$, $g((\overline{\nabla}_X J)(Y), Y) = 0$, $g((\overline{\nabla}_X J)(Y), JX) = 0$, $g((\overline{\nabla}_X J)(Y), JY) = 0$, where we have used $(\overline{\nabla}_X J)(Y) = -(\overline{\nabla}_Y J)(X)$ and $(\overline{\nabla}_X J)(JY) = -J(\overline{\nabla}_X J)(Y)$, which is consequence of $(\overline{\nabla}_X J)(X) = 0$. This proves that $(\overline{\nabla}_X J)(Y) \in \overline{\mu}$. $(\overline{\nabla}_X J)(Y)$ is a unit vector follows directly from Lemma (cf.[1], p. 760).

2. We shall be concerned with 2-dimensional totally real submanifold M of S^6 for which $h(X,Y) \in J$TM, $X,Y \in \mathcal{X}(M)$. With the help of (1.1), we obtain

$$(\overline{\nabla}_X J)(Y) = \nabla^\perp_X JY - A_{JY}X - J\nabla_X Y - Jh(X,Y).$$

Since $(\overline{\nabla}_X J)(Y) \in \overline{\mu}$, and by our assumption $Jh(X,Y)$ is tangent to M, on equating tangential and normal components in above equation we obtain

$$\nabla^\perp_X JY = J\nabla_X Y + (\overline{\nabla}_X J)(Y), \quad h(X,Y) = JA_{JY}X, \quad X,Y \in \mathcal{X}(M). \tag{2.1}$$

From the second equation in (2.1) using the symmetry of $h(X,Y)$, we get $A_{JY}X = A_{JX}Y$ and consequently we obtain

$$g(h(X,Y),JZ) = g(h(Y,Z),JX) = g(h(X,Z),JY). \tag{2.2}$$

Theorem 2.1. *Let M be a 2-dimensional totally real submanifold of constant Gaussian curvature of S^6 with $h(X,Y) \in J\mathrm{TM}$. Then the Gaussian curvature of M is 1, that is, M is totally geodesic.*

Proof. Let $UM = \{X \in \mathrm{TM} : \|X\| = 1\}$ be the unit tangent bundle of M. Consider the function $f : UM \to R$ defined by $f(X) = g(h(X,X),JX)$, which is clearly smooth. We claim that if f is constant, then M has Gaussian curvature 1. Since $f(-X) = -f(X)$, if f is constant, then $f = 0$, and consequently $g(h(X,X),JX) = 0$, $X \in TJM$. For any orthonormal frame $\{X,Y\}$ of M, we shall get $g(h(X,X),JX) = 0$, $g(h(Y,Y),JY) = 0$, $g(h(\frac{X+Y}{\sqrt{2}}, \frac{X+Y}{\sqrt{2}}), J(\frac{X+Y}{\sqrt{2}})) = 0$ and $g(h(\frac{X-Y}{\sqrt{2}}, \frac{X-Y}{\sqrt{2}}), J(\frac{X-Y}{\sqrt{2}})) = 0$. Using (2.2) in these equations we get $g(h(X,Y),JX) = 0$ and $g(h(X,Y),JY) = 0$. Since $h(X,Y) \in J\mathrm{TM}$ and $\{JX, JY\}$ is an orthonormal frame of $J\mathrm{TM}$, we get $h(X,X) = 0$, $h(Y,Y) = 0$, $h(X,Y) = 0$, that is, we get that M is totally geodsic and hence by (1.2), the Gaussian curvature of M is 1

Suppose f is not a constant. The unit tangent bundle UM being compact, f attains maximum. Suppose f attains maximum at e_1. Then it is known that $g(h(e_1,e_1),JY) = 0$ for $Y \perp e_1$, $Y \in UM$ (cf.[1]). Put $h(e_1,e_1) = \lambda Je_1$, where λ is a smooth function on M. Choose e_2 such that $\{e_1,e_2\}$ is a local orthonormal frame of M. Then using (2.2) we have the following expressions

$$h(e_1,e_1) = \lambda Je_1, \quad h(e_2,e_2) = \mu Je_1 + \nu Je_2, \quad h(e_1,e_2) = \mu Je_2, \tag{2.3}$$

where μ, ν are smooth fucntions on M.

From the structure equations of M we have the following local equations

$$\nabla_{e_1} e_1 = ae_2, \quad \nabla_{e_2} e_2 = be_1, \quad \nabla_{e_1} e_2 = -ae_1, \quad \nabla_{e_2} e_1 = -be_2, \tag{2.4}$$

where a, b are smooth functions.

Taking (1.3) in form of the following two equations

$$(\overline{\nabla}_{e_1} h)(e_1, e_2) = (\overline{\nabla}_{e_2} h)(e_1, e_1), \quad (\overline{\nabla}_{e_2} h)(e_1, e_2) = (\overline{\nabla}_{e_1} h)(e_2, e_2)$$

and using (2.1), (2.3) and (2.4), on equating the components we obtain

$$e_1 \cdot \mu = a\nu - b(\lambda - 2\mu), \quad e_2 \cdot \lambda = a(\lambda - 2\mu), \quad (\lambda - \mu)(\overline{\nabla}_{e_1} J)(e_2) = 0 \tag{2.5}$$

$$e_2 \cdot \mu - e_1 \cdot \nu = -b\nu + 3a\mu, \quad \nu(\overline{\nabla}_{e_1} J)(e_2) = 0. \tag{2.6}$$

Since from Lemma 1.1, $(\overline{\nabla}_{e_1} J)(e_2)$ is unit vector, we get that $\lambda = \mu$ and $\nu = 0$. The Gaussian curvature K of M is then found from (1.2) and (2.3) as $K = 1 + \lambda^2 - \lambda^2 = 1$. This proves the Theorem.

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