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NEAT AND NEAT-HIGH EXTENSIONS

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Abstract. In the present paper, we find conditions that change a neat-high subgroup into a pure-high subgroup and a neat subgroup, to a pure subgroup. It is shown that a neat-high extension can be transformed into a pure-high extension and a neat extension, to a pure extension. Furthermore, splitting conditions for a neat exact sequence are obtained.

1. Introduction

A subgroup A of a group G is called pure in G if $nA = A \cap nG$, for all natural numbers n. This concept was generalized by Honda [4]. He defined, a subgroup A of a group G is neat in G, if $pA = A \cap pG$, for all prime numbers p. This is equivalent to $mA = A \cap mG$, for all square-free natural numbers m. Every pure subgroup of a group G is neat in G, but the converse is not always true. Every direct summand of a group G is pure in G and also neat in G. A sequence $O \rightarrow A \xrightarrow{f} G \rightarrow C \rightarrow O$ is called pure exact if $f(A)$ is a pure subgroup of G and neat exact if $f(A)$ is a neat subgroup of G. Let B be a subgroup of G, the exact sequence $O \rightarrow A \rightarrow G \rightarrow C \rightarrow O$ is called a B-pure-high extension, if A is maximal disjoint from B and $(A+B)/B$ is pure in G/B and a B-neat-high extension, if A is maximal disjoint from B and $(A+B)/B$ is neat in G/B. All groups considered in this paper are abelian.

In grneral, we adopt the notations used in [2].

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2. Main Results

First we find conditions for a B-neat-high subgroup of a p-group G to become a B-pure-high subgroup of G. In this direction we prove the following:

Theorem 1. *If A is a B-neat-high subgroup of a p-group G such that $G/B = H/B \oplus K/B$, where H/B is a direct sum of cyclic groups of order p^n and K/B is a direct sum of cyclic groups of order p^{n+1}, then A is a B-pure-high subgroup of G.*

Proof. Let $\{H_i/B : i \in I\}$ denote the cyclic summands of H/B having order p^n and $\{K_j/B : j \in J\}$, those of K/B having order p^{n+1}. It is clear that for $K \geq n+1$, $p^k(G/B) = 0$ and for such K,

$$p^k(G/B) \cap (A+B)/B \;=\; p^k((A+B)/B)$$

Thus we now assume that

$$p^m(G/B) \cap (A+B)/B \;=\; p^m((A+B)/B)$$

for some m, such that $1 \leq m < n$. We prove that it holds for p^{m+1} also, for this purpose let $p^{m+1}(x+B \in (A+B)/B$. for some $(x+B) \in G/B$. It follows that $p^{m+1}(x+B) = (a'+B)$ for some $(a'+B) \in (A+B)/B$. But by assumption, for $p^m(g+B) = (a'+B)$, $(g+B) \in G/B$ there exists an element $(a+B) \in (A+B)/B$ such that $p^m(a+B) = (a'+B)$. Thus

$$p^{m+1}(x+B) \;=\; p^m(a+B) \tag{1}$$

$\Rightarrow p^m(p(x+B)-(a+B)) \;=\; 0 \Rightarrow (p(x+B)-(a+B)) \in G/B[p^m]$

If (x_i+B) and (a_i+B) denote the i^{th} coordinates of $(X+B)$ and $(a+B)$ respectively, then it follows that $(p(x_i+B)-(a_i+B)) \in H_i/B[p^m] = p^{n-m}(H_i/B) \subseteq p(H_i/B) \Rightarrow (a_i+B) \in p(H_i/B)$, for each $i \in I$. Similarly, $(a_j+B) \in p(K_j/B)$ for each $j \in J$. Consequently, there exists $(a+B) \in p(G/B)$, let $(a+B) = p(g+B)$, for $(g+B) \in G/B$. By neatness of $(A+B)/B$ in G/B, we are guaranteed an element $(a' + B) \in (A+B)/B$ satisfying $p(g+B) = (a+B) = p(a' + B)$. But then (1) gives

$$p^{m+1}(x+B) \;=\; p^m(a+B) \;=\; p^{m+1}(a'+B) \in p^{m+1}((A+B)/B)$$

Thus $(A+B)/B$ is pure-high in G/B and A is a B-pure-high subgroup of G.

In case of neat subgroups we have the following.

Theorem 2. *If A is a neat subgroup of a p-group G such that $G = H \oplus K$, where H is a direct sum of cyclic groups of order p^n and K is a direct sum of cyclic groups of order p^{n+1}, then A is a pure subgroup of G.*

Proof. The proof runs on similar lines as that of theorem 1.

Next, we find conditions under which the neatness of $(A+B)/B$ in G/B reduces to purity of $(A+B)/B$ in G/B.

Lemma 1. *If n is a square-free natural number and A and B are subgroups of G such that $n(A+B) = 0$, then $(A+B)/B$ is neat in G/B if and only if $(A+B)/B$ is a direct summand of G/B.*

Proof. The factor group $\frac{G/B}{<(A+B)/B,\, n(G/B)>}$ is bounded and hence by theorem 17.2 of [2] is a direct sum of cyclic groups. Let $\frac{G/B}{<(A+B)/B,\, n(G/B)>} = \oplus_{i \in S} < \overline{x}_i >$, where S is the set of all square-free natural numbers and $< \overline{x}_i >$ is cyclic of order n_i. Define the natural homomorphism $f : G/B \rightarrow \underset{i \in S}{\oplus} < \overline{x}_i >$ in such a way that for each $i \in S$ we choose $(x_i + B) \in G/B$ so that $f(x_i + B) = \overline{x}_i$, then

$$f(n_i(x_i+B)) = n_i\,\overline{x}_i = 0 \;\Rightarrow\; n_i(x_i+B) \in < (A+B)/B,\; n(G/B) >$$

Let $n_i(x_i + B) = (a_i + B) + n(g_i + B)$, for some $(a_i + B) \in (A+B)/B$ and $(g_i + B) \in G/B$. Since $n\overline{x}_i = 0$, it follows that n_i divides n and hence n_1 is itself square-free for all i. Now,

$$(a_i + B) \;=\; n_i((x_i+B) - n/n_i\,(g_i+B)) \tag{1}$$

Neatness of $(A+B)/B$ in G/B implies that there exists $(a_i'+B) \in (A+B)/B$ such that $n_i(a_i'+B) = a_i + B$. If we set $y_i + B = (x_i + B) - (a_i' + B)$, then with the help of relation (1)

$$n_i(y_i + B) = (a_i + B) + n(g_i + B) - n_i(a_i' + B) = n(g_i + B)$$

Furthermore, $f(y_i + B) = f(x_i + B) = \overline{x}_i$ (2)

Define, $L =< n(G/B), \ldots, y_i + B \ldots >$. We prove that $G/B = (A+B)/B \oplus L$.

If $x \in (A + B)/B \cap L$, then

$$x = \sum_{i \in S} m_i(y_i + B) + (ng + B) \in (A + B)/B$$

and with the help of relation (2), $f(x) = \sum_{i \in S} m_i \overline{x}_i = 0$ implies n_i divides m_i. Now,

$$n_i(y_i + B) = n(g_i + B) \in n(G/B) \Rightarrow m_i(y_i + B) \in n(G/B)$$

$\Rightarrow x \in n(G/B)$. Consequently,

$$x \in (A + B)/B \cap n(G/B) = n((A + B)/B) = 0$$

and hence $(A + B)/B \cap L = 0$.

Now if $(g + B) \in G/B$, then definition of f and relation (2) implies that

$$f(g + B) = \sum_{i \in S} m_i' \overline{x}_i = f \sum_{i \in S} m_i'(y_i + B)$$

$$\Rightarrow f((g + B) - \sum_{i \in S} m_i'(y_i + B) = 0$$

$$\Rightarrow ((g + B) - \sum_{i \in S} m_i'(y_i + B)) \in < (A + B)/B, n(G/B) >$$

$$\Rightarrow (g + B) - \sum_{i \in S} m_i'(y_i + B) = (a + B) + n(g' + B)$$

for some $(a + B) \in (A + B)/B$ and $(g' + B) \in G/B$

$$\Rightarrow (g + B) \in < (A + B)/B, L >$$

With the help of lemma 1, the proof of the following theorem is clear.

Theorem 3. *If n is a square-free natural number, a B-neat-high extension $0 \to A \to G \to C \to 0$ is a B-pure-high extension if $n(A+B)=0$.*

If we focus our attension to neat subgroups we have an analogue of the well known theorem that a bounded subgroup of a group G is pure if and only if it is a direct summand of G.

Lemma 2. *If n is a square-free natural number and A a subgroup of a group G such that $nA=0$, then A is a neat subgroup of G if and only if A is a direct summand of G.*

Proof. Follows on similar lines as that of lemma 1.

The proof of the following theorem is clear.

Theorem 4. *If n is a square-free natural number, then the neat extension $0 \to A \to G \to B \to 0$ splits if $nA=0$.*

Theorem 5. *If n is a square-free natural number and A is a nG-high subgroup of G, then the sequence $0 \to A \to G \to B \to 0$ is splitting neat exact.*

Proof. If A is a subgroup of G which is maximal with respect to the property $A \cap nG = 0$, then A is neat in G, (see [3]), that is, $mA = A \cap mG$ for all square-free natural numbers m. In particular, if $m = n$ then $nA = 0$ and theorem 4 completes the proof.

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