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LIFTING AND PROJECTING BETWEEN RANDERS AND RIEMANNIAN SPACES

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§1. Introduction.

Let M_n be an n-dimensional differentiable manifold and TM_n its tangent bundle, which is a $2n$-dimensional differentiable manifold. The lifting process of tensor fields and connections from the base manifold M_n to its tangent bundle TM_n has been studied by many authors ([10], [11], ..., etc.). Although great attention has been paid to this process, the process of raising the dimension of the base manifold by *one* has attracted few mathematicians. However, many problems in relativity, mechanics, ..., etc., depend to a great extent on this notion. One of the most interesting works in this direction is that of A. Lichnerowicz [4]. In his study he has introduced a notion of projecting (resp. lifting) Lagrangian functions defined on M_{n+1} (resp. M_n) onto M_n (resp. to M_{n+1}).

The purpose of the present paper is to introduce a notion of lifting and projecting vector fields from one space to another with dimensions different by *one*. Having done so, the mutual relations of the geometric objects (metric tensors, geodesics, connections, curvature tensors, Jacobi fieldss, ..., etc.) on both manifolds, one of them is Finsler and the other is Riemannian, are established.

In §2, we give a brief survey for the notions and results to be used latter on. We follow the approach of Lichnerowicz [4] in:
1) Lifting a Randers space (M_n, L) to a Riemannian space $(M_{n+1},\mathcal{L})$ such that the projection on M_n of extremals of $\mathcal{L}$ are extremals of L.
2) Projecting the Riemannian space $(M_{n+1},\mathcal{L})$ noto the space (M_n, L) such that

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every extremals of $\mathcal{L}$, under certain condition, is projected on an extremal of L.

§3 is devoted to the relation between connection coefficients of the two spaces $(M_{n+1}, \mathcal{L})$ and $(M_n, {}^\circ L)$.

The study of the lifting process is the object of §4. A necessary condition for the lifting of a Jacobi field defined in $(M_n, {}^\circ L)$ to be a Jacobi field in $(M_{n+1}, \mathcal{L})$ is derived. The well known Morse index theorem is then applied to our obtained results.

§5 is concerned with the study of projection process. A necessary condition for the projection of a Jacobi field defined in $(M_{n+1}, \mathcal{L})$ to be a Jacobi field in $(M_n, {}^\circ L)$ is given.

In the last section, §6, we conclude our study with two significant examples.

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§2. Preliminaries.

By a differentiable manifold M_n we will always mean a C^∞ connected manifold of finite dimension n covered by a system of coordinate neighborhoods $\{U; x^i\}$ where U denotes a neighborhood and x^i denote the local coordinates in U, with Latin indices $i, j, k, \ldots$ taking on values in the range $1, 2, \ldots, n$. All the geometric objects we are interested with are assumed to be smooth. For every bundle $E \to N$, we will always denote E_x by the fibre over x in N.

Let ∇ be a Riemannian (Levi-Civita) connection on M_n, by a Jacobi field we mean a vector field J along a geodesic c in M_n such that:

$$D^2 J/dt^2 + R(V, J)V = 0$$

where $V = dc/dt$, R is the Riemannian curvature tensor and D/dt is the covariant derivative operator associated with the Riemannian connection.

By a Randers space we mean a Finsler space (M_n, L) with a fundamental function $L(x,\dot{x}) = {}^\circ L(x,\dot{x}) + {}^\circ\beta(x,\dot{x})$ where ${}^\circ L(x,\dot{x}) = \sqrt{g_{ij}(x^k)\dot{x}^i\dot{x}^j}$ [1], g_{ij} is a Riemannian metric tensor and ${}^\circ\beta(x,\dot{x}) = b_i(x^k)\dot{x}^i$.
For more details on Randers space we refer to [2], [5] and [9].

Appealing to the same hypothesis as in [4], in the case of a charged particle in M_n; a global potential B does exist such that the trajectories of this particle can be defined as the extremals of the integral associated with the function $L(x,\dot{x})$ given above. It is therefore possible to interpret the trajectories of the charged particle in M_n as the projection of geodesics in the Riemannian manifold of dimension $n+1$ such that $G_{oo} = 0$ in the adapted coordinates "i.e. the natural coordinates (x°, x^i) in which the metric tensor G is independent of the variable x° and the trajectories of the vector field ξ which generates the global 1-parmeter group of the local transformations of the manifold has the form $\xi^i = 0$ and $\xi^\circ = 1$".

Let $\mathcal{L}^2(x^i, \dot{x}^\alpha) = G_{\alpha\beta}(x^k)\dot{x}^\alpha\dot{x}^\beta$, where $\mathcal{L}$ is homogeneous function of degree 1 in $\dot{x}$ and $G_{\alpha\beta}$ are the components of a metric tensor on M_{n+1} which is independent of the variable x° in the adapted coordinates, with Greek indices $\alpha, \beta, \ldots$ taking on values in the range $0, 1, 2, \ldots, n$.

We shall assume that $\partial_{\dot{o}\dot{o}}\mathcal{L} \neq 0$ [2]. As $\partial_o\mathcal{L} = 0$, then we have, using Euler-Lagrange equations, $d(\partial_{\dot{o}}\mathcal{L})/dt = 0$. Thus

$$\partial_{\dot{o}}\mathcal{L} = h \quad (1) \text{ or } \partial_{\dot{o}}\mathcal{L}^2 = 2h\mathcal{L}. \quad (2)$$

But since

$$\mathcal{L}^2 = G_{oo}(\dot{x}^\circ)^2 + 2G_{oi}\dot{x}^\circ\dot{x}^i + G_{ij}\dot{x}^i\dot{x}^j, \quad \text{with } G_{oo} \neq 0$$

1) We will always use the summation convention, i.e. repeated indices imply summation.

2) $\partial_\alpha\mathcal{L} = \partial\mathcal{L}/\partial x^\alpha$, $\partial_{\dot{\alpha}}\mathcal{L} = \partial\mathcal{L}/\partial\dot{x}^\alpha, \ldots, \partial_{\dot{\alpha}\dot{\beta}}\mathcal{L} = \partial^2\mathcal{L}/\partial\dot{x}^\alpha\partial\dot{x}^\beta$

we get

$$\mathcal{L}^2 = (G_{oo}\dot{x}^o + G_{oi}\dot{x}^i)^2/G_{oo} + (G_{ij} - G_{oi}G_{oj}/G_{oo})\dot{x}^i\dot{x}^j. \tag{3}$$

Therefore, equation (2) can be written in the form

$$\partial_{\dot{o}}\mathcal{L}^2 = 2h\mathcal{L} = 2(G_{oo}\dot{x}^o + G_{oi}\dot{x}^i). \tag{4}$$

As the extremals of $\mathcal{L}$ are charaterized by the condition $\partial_{\dot{o}}\mathcal{L} = h$, and noting that $G_{oo} \neq 0$, therefore, by solving equation (1) for $\dot{x}^o$ we get $\dot{x}^o = \phi(x^i, x^j, h)$, where ϕ is a homogeneous function of degree 1 in $\dot{x}^j$ and depends on h.

Using equation (4), equation (3) can be written as:

$$\mathcal{L}^2 = h^2\mathcal{L}^2/G_{oo} + @^2$$

where

$$@^2 = g_{ij}\dot{x}^i\dot{x}^j \quad \text{and} \quad G_{ij} - G_{oi}G_{oj}/G_{oo},$$

Hence

$$@^2 = (1 - h^2/G_{oo})\mathcal{L}^2,$$

and then

$$@ = \mathcal{L}\sqrt{1 - h^2/G_{oo}}. \tag{5}$$

From homogenety of $\mathcal{L}$, we have

$$\dot{x}^k\partial_{\dot{k}}\mathcal{L} = \mathcal{L} - h\dot{x}^o, \quad \text{along an extremal of} \quad \mathcal{L}. \tag{6}$$

Using equation (4), the variable $\dot{x}^o$ can be expressed as a function of both $\mathcal{L}$ and $(x^i, \dot{x}^j)$ as :

$$\dot{x}^o = (h\mathcal{L} - G_{oi}\dot{x}^i)/G_{oo}. \tag{7}$$

Thus in equation (6), the quantity $\dot{x}^k\partial_{\dot{k}}\mathcal{L}$ can be expressed by a function $^*L$ of the variable $(x^i, \dot{x}^j, h)$ as:

$$^*L(x^i, \dot{x}^j, h) = \mathcal{L}(x^i, \dot{x}^j, \phi(x^i, \dot{x}^j, h)) - h\phi(x^i, \dot{x}^j, h). \tag{8}$$

Also, we have

$$\partial_k {}^*L = \partial_k \mathcal{L} + \partial_{\dot{o}} \mathcal{L}\, \partial_k \phi - h \partial_k \phi.$$

Using equation (1), we have along an extremal of $\mathcal{L}$, that:

$$\partial_k {}^*L = \partial_k \mathcal{L}.$$

Now let

$$I = \int_{t_0}^{t_1} \mathcal{L}\, dt = \int_{\widehat{AB}} \partial_{\dot{\alpha}} \mathcal{L}\, dx^\alpha, \text{ where } \widehat{AB} \text{ is an extremal of } \mathcal{L},$$
$$= \int_{\widehat{AB}} (\partial_k \mathcal{L}\, dx^k + h\, dx^\circ) = \int_{\widehat{ab}} \partial_k {}^*L\, dx^k + h(x^\circ(B) - x^\circ(A)),$$

where $\widehat{ab}$ is an arc in M_n.

Thus the extremals of $\mathcal{L}$ such that $\partial_{\dot{o}} \mathcal{L} = h$ project on exteremals of

$$J = \int_{\widehat{ab}} \partial_k {}^*L\, dx^k = \int_{t_0}^{t_1} {}^*L\, dt.$$

Therefore, the projection of extremals of $\int \mathcal{L}(x, x) dt$ which correspond to the value h on M_n are extremals of the integral

$$\int {}^*L(x^i, \dot{x}^j, h) dt \tag{9}$$

where h has a chosen value.

In conclusion, we have for any homogeneous function $\mathcal{L}\ (x^i, \dot{x}^\alpha)$ of degree 1 with respect to $\dot{x}^\alpha$ such that $\partial_{\dot{o}\dot{o}} \mathcal{L} \neq 0$, the extremals on M_{n+1} of the integral $\int \mathcal{L}(\dot{x}, \dot{x}) dt$ which correspond to the value h given by $\partial_{\dot{o}} \mathcal{L} = h$, will be projected on M_n via the extremals of the integral (9), where h has the same chosen value and ${}^*L$ is given by (δ).

Moreover, using equations (5) and (7), equation (8) can be rewritten in the form

$${}^*L(x^i, \dot{x}^j, h) = \sqrt{(1 - h^2/G_{oo}) g_{ij} \dot{x}^i \dot{x}^j} + h(G_{oi}/G_{oo}) \dot{x}^i. \tag{10}$$

Now, comparing this function with the initial function L of the Randers space, we notice that the quadratic form $g_{ij} \dot{x}^i \dot{x}^j$ is multiplied by a factor $(1 - h^2/G_{oo})$

which depends in general on the parameter h and x^i by the intermediation of G_{oo}; which is not the case for L. We can avoid this difficulty by choosen G_{oo} = constant, and by defining a new constant μ which modify the numerical value of G_{oo} if it is necessary, by

$$\mu h = \sqrt{1 - h^2/G_{oo}}.$$

In this case, the extremals of L are also the extremals of

$$\sqrt{1 - h^2/G_{oo}}L(x, \dot{x}) = \sqrt{(1 - h^2/G_{oo})g_{ij}\dot{x}^i\dot{x}^j} + \mu h b_i \dot{x}^i. \tag{11}$$

Identifying equation (11) with ${}^*L(x^i, \dot{x}^j, h)$, we see that the symbol g_{ij} stands for the same quantities and also $G_{oi} = \mu G_{oo} b_i$. Also, we have

$$G_{oo} = \text{constant}, \quad G_{oi} = \mu G_{oo} b_i \quad \text{and} \quad G_{ij} = g_{ij} + \mu^2 G_{oo} b_i b_j.$$

The Riemannian metric sought can be written as:

$$\begin{aligned} d\sigma^2 &= \mathcal{L}^2(x^i, dx^\alpha) = G_{\alpha\beta}(x^i)dx^\alpha\, dx^\beta \\ &= g_{ij}\, dx^i\, dx^j + G_{oo}(dx^o + \mu b_i\, dx^i)^2. \end{aligned} \tag{12}$$

Now, for simplicity, we take $G_{oo} = 1$ and $h = 1/\sqrt{2}$, then $\mu = 1$.

Therefore, we get

i) $G_{oo} = 1$, $G_{oi} = b_i$, $G_{ij} = g_{ij} + b_i b_j$,

ii) $L = {}^*L/\sqrt{2}$, i.e. L and ${}^*L$ have the same extremals,

iii) $d\sigma^2 = g_{ij}\, dx^i\, dx^j + (dx^o + b_i dx^i)^2$
$\quad = ds^{o2} + (dx^o + b_i\, dx^i)^2$

or

$$\mathcal{L} = {}^oL^2 + \beta^2$$

where

$${}^oL^2 = g_{ij}\dot{x}^i\dot{x}^j \quad \text{and} \quad \beta(x^i, \dot{x}^\alpha) = b_\alpha(x^i)\dot{x}^\alpha = \dot{x}^o +^o \beta. \tag{13}$$

It is clear that, the inverse metric $(G^{\alpha\beta})$ of the metric tensor $G = (G_{\alpha\beta})$ on M_{n+1} is given by:

$$G^{oo} = 1 + b^2, \quad G^{oi} = -b^i \quad \text{and} \quad G^{ij} = g^{ij}, \tag{14}$$

where

$$b^2 = b_i b^i.$$

Now, it is important to compute the vatiation of x° along a trajectory in M_n. The calculation can be made in the general case for a geodesic of $\mathcal{L}^2$ corresponding to the value h.

From equations (5) and (7), we have

$$\dot{x}^\circ = (h/G_{\circ\circ})\sqrt{g_{ij}\dot{x}^i\dot{x}^j/(1-h^2/G_{\circ\circ})} - (G_{\circ i}/G_{\circ\circ})\dot{x}^i.$$

But as

$$G_{\circ i} = \mu G_{\circ\circ} b_i \quad \text{and} \quad \mu h = \sqrt{1-h^2/G_{\circ\circ}}\ ,$$

then

$$\dot{x}^\circ = (1/\mu G_{\circ\circ})\sqrt{g_{ij}\dot{x}^i\dot{x}^j} - \mu b_i \dot{x}^i.$$

Consequently

$$dx^\circ = (1/\mu G_{\circ\circ})ds^\circ - \mu b_i\, dx^i. \tag{15}$$

Hence, for $G_{\circ\circ} = 1$ and $h = 1/\sqrt{2}$, we have

$$dx^\circ = ds^\circ - b_i\, dx^i. \tag{16}$$

Integrating, we get

$$x^\circ = \int_{t_0}^{t_1} ds^\circ - \int_{t_0}^{t_1} b_i dx^i + c, \tag{17}$$

where t denotes an arbitrary parameter.

Choosing the constant c along every trajectory in M_n, we can evaluate the function $x^\circ(t)$ from equation (17). The expressions for $x^\circ(t)$ and $x^i(t)$ give a parametric representation of a geodesic of (13) iii).

Therefore, we have

Theorem 1. [4]. *The trajectories in M_n can be obtained as follows: given a constant c, we consider a manifold M_{n+1} homeomorphic to $M_n \times R$ and endowed with the Riemannian metric (13) iii), where x° is the abscisa of a point on*

R. The considered trajectories are the projections on M_n of geodesics in the Riemannian manifold M_{n+1}.

Conversly, if for any given point of the trajectory, we associate the point of M_{n+1} defined by (17) whose projection on M_n is the given point, then this point describes a geodesic in M_{n+1} satisfying the preceding conditions.

Let c be a geodesic in M_n. The geodesic in M_{n+1} obtained using the method mentioned above will be denoted by c_ϱ and called the canonical lifting of c to M_{n+1}. If $c(t) = (x^i(t))$, then $c_\varrho(t) = (x^\circ(t), x^i(t))$.

We have to point out that the canonical lifting is taken over the submanifold of TM_{n+1} defined locally by $\partial_{\dot{o}}\mathcal{L} = 1/\sqrt{2}$. As $\mathcal{L} = \sqrt{{}^\circ L^2 + \beta^2}$ and $h = 1/\sqrt{2}$, we get $\beta = {}^\circ L$. Thus the canonical lifting of $c(t)$ defined by the n-functions $x^i(t)$ is such that $\dot{x}^\circ(t) = {}^\circ L(x^i(t), \dot{x}^i(t)) - b_i(x^k(t))\dot{x}^i(t)$. On the other hand, we can assume that $t = s^\circ$ with $ds^{\circ 2} = g_{ij}dx^i dx^j$. Therefore ${}^\circ L = 1$ and $\dot{x}^\circ = 1 - b_i\dot{x}^i$.

The sections $x^\circ =$ constant will be denoted by W_n. These are differentiable manifolds of dimension n which are locally diffeomorphic to M_n.

In the rest of this section, we relate the geodesics of both ${}^*L$ and ${}^\circ L$.

Lemma 1. *A Randers space (M_n, L) and a Riemannian space $(M_n, {}^\circ L)$, where $L = {}^\circ L + {}^\circ\beta$, have the same geodesics if, and only if, the 1-form $b_i dx^i$ is closed.*

Proof. From the definition of a Randers space, we have

$$L(x, \dot{x}) = {}^\circ L(x, \dot{x}) + {}^\circ\beta(x, \dot{x})$$

thus

$$P_k(L) = P_k({}^\circ L) + (\partial_j b_k - \partial_k b_j)\dot{x}^j.$$

Therefore

$$P_k(L) = P_k({}^\circ L) \quad \text{if, and only if,} \quad d(b_j(x^i)dx^j) = 0,$$

Where P_k is the Euler-Lagrange operator, i.e.

$$P_k(L) = d(\partial_{\dot{k}} L)/dt - \partial_k L.$$

The extremals of both $^*L$ and $L =^\circ L +^\circ \beta$, are the projections on M_n of exyremals of $\pounds$ which satisfy the condition $\partial_{\dot{o}}\pounds = 1/\sqrt{2}$.

If the 1-form $b_i dx^i$ is closed, then the fundamental functions $^\circ L$ and L both have the same extremals (cf. lemma 1), and so the fundamental functions $^*L$ and $^\circ L$ both have the same extremals.

§3. Connection coefficients of the space $(M_{n+1}, \pounds)$.

Let $q^\lambda_{\alpha\beta}$ be the connection coefficients of the Riemannian connection on M_{n+1}.

We can get by direct computation that:

$$\begin{aligned} q^k_{ij} &= \{{k \atop ij}\} + \overset{\circ}{A}{}^k_{ij}, \quad q^\circ_{ij} = b_{(ij)} + (b_{[jh]}\, b_i + b_{[ih]}\, b_j) b^h, \\ q^k_{\circ\circ} &= 0 = q^\circ_{\circ\circ}, \quad q^k_{\circ j} = q^k_{j\circ} = g^{hk} b_{[hj]}, \quad \text{and} \\ q^\circ_{\circ j} &= b_{[jh]} b^h, \end{aligned} \tag{18}$$

where $\{{K \atop ij}\}$ are the Christoffel symbols of the second kind with respect to the initial Riemannian space $(M_n, ^\circ L)$,

$$\begin{aligned} b_{ij} &= \partial_k b_j - \{{r \atop jk}\} b_r, \\ b_{(jk)} &= \frac{1}{2}(b_{jk} + b_{kj}) \\ b_{[jk]} &= \frac{1}{2}(\partial_j b_k - \partial_k b_j) \\ \text{and} \qquad \overset{\circ}{A}{}^k_{ij} &= g^{hk}(b_{[hi]} b_j + b_{[hj]} b_i). \end{aligned}$$

It is clear that:

Lemma 2. *If the 1-form $b_i dx^i$ is closed, then the connection coefficients of the Riemannian connection on M_{n+1} are given by*

$$q^k_{ij} = \{{k \atop ij}\}, \quad q^\circ_{ij} = b_{ij}, \tag{19}$$

and the other components vanish.

§4. The lifting.

Let us denote by ∇' the Riemannian connection on M_{n+1} and ∇ the Riemannian connection on $(M_n, {}^\circ L)$.

Let c be a geodesic in M_n, and c_ϱ be its canonical lift to M_{n+1}. Given any vector field $X = X^i\partial_i$ along c, its canonical lift along c_ϱ is defined by $X_\varrho = X^\alpha\partial_\alpha = X^\circ\partial_\circ + X^i\partial_i$, where $X^\circ = {}^\circ L(X) - w(X)$; $w = b_i dx^i$. We may write $X_\varrho = X_\circ + X$ where $X_\circ = X^\circ\partial_\circ$.

We shall denote by $X, Y, \ldots$ the vector fields along c in M_n, and $X_\varrho, Y_\varrho, \ldots$ their liftings along c_ϱ.

It is clear that: the lifting of the sum of two vector fields along C is equal to the sum of their lifting along c_ϱ.

That is:

$$(X+Y)_\varrho = X_\varrho + Y_\varrho. \qquad (a)$$

We shall establish the following formulas which are to be needed in the sequel.

Since

$$[X_\varrho, Y_\varrho] = [X, Y] + (X \cdot Y^\circ - Y \cdot X^\circ)\partial_\circ$$

where $X \cdot Y^\circ$ denotes the Lie derivative of the function Y° with respect to X.

But as:

$$[X, Y]_\varrho = [X, Y]_\circ + [X, Y],$$

we have

$$[X_\varrho, Y_\varrho] = [X, Y]_\varrho + [X, Y] \qquad (b)$$

where

$$[X, Y]' = (X.Y^\circ - Y.X^\circ)\partial_\circ - [X, Y]_\circ.$$

Now,

$$\nabla'_{X_\varrho} Y_\varrho = \nabla'_X Y_\varrho + \nabla'_{X_\circ} Y_\varrho.$$

As,

$$\nabla'_{X_\circ} Y_\varrho = 0, \quad \nabla'_X Y_\circ = (X.Y^\circ)\partial_\circ \quad \text{and} \quad \nabla'_X Y = \nabla_X Y + b_{(ij)} X^i Y^j \partial_\circ$$

then

$$\nabla'_{X_\varrho} Y_\varrho = \nabla_X Y + X^i(\partial_i Y^\circ + b_{(ij)} Y^j). \tag{20}$$

Also

$$(\nabla_X Y)_\varrho = \nabla_X Y + (\nabla_X Y)_\circ.$$

Therefore the relation between $\nabla'_{X_\varrho} Y_\varrho$ and the lifting of $\nabla_X Y$ is given by

$$\nabla'_{X_\varrho} Y_\varrho = (\nabla_X Y)_\varrho + C(X,Y), \tag{c}$$

where

$$C(X,Y) = (X.Y^\circ + b_{(ij)} X^i Y^j)\partial_\circ - (\nabla_X Y)_\circ.$$

Now, let D'/dt and D/dt be the covariant derivative operators associated with the Riemannian connections ∇' on M_{n+1} and ∇ on M_n respectively. Let X be a vector field along c and X_ϱ be its canonical lift along c_ϱ. From equation (20), we have

$$D'X_\varrho/dt = DX/dt + (dX^\circ/dt + b_{(ij)}\dot{x}^i X^j)\partial_\circ.$$

As,

$$dX^\circ/dt = -[\partial_j b_i)\dot{x}^j X^i + b_i\, dX^i/dt],$$

thus

$$D'X_\varrho/dt = DX/dt - (b_i dX^i/dt + b_r \{ {r \atop ij} \} \dot{x}^i X^j)\partial_\circ.$$

But

$$(DX/dt)_\varrho = DX/dt + (DX/dt)_\circ,$$

therefore, we have

$$D'X_\varrho/dt = (DX/dt)_\varrho -^\circ L(DX/dt)\partial_\circ. \tag{d}$$

Let us denote by R' (respectively R) the curvature tensor of the connection ∇' (respectively ∇).

From formulas (a), (b), (c) and the definition of the curvature, we have

Proposition 1.

$$R'(X_\varrho, Y_\varrho)Z_\varrho = (R(X,Y)Z)_\varrho + \tilde{A}(X,Y)Z$$

where

$$\tilde{A}(X,Y)Z = \{C(Y, \nabla_X Z) + \nabla'_y C(X,Z) - (X/Y)\} + C([X,Y],Z)$$

and (X/Y) means the same terms repeated with X and Y interchanged.

Using formula (a) and assuming that the scalar 1-form $b_i dx^i$ is closed, then we get, by direct calculation that:

$$\tilde{A}(X,Y)Z = {}^*r\, \partial_\circ,$$

where

$$*r = {}^\circ L(\nabla_X \nabla_y Z) -^\circ L(\nabla_y \nabla_X Z) -^\circ L(\nabla_{[X,Y]} Z).$$

Thus the relation between $R'(X_\varrho, Y_\varrho)Z_\varrho$ and the lifting of $R(X,Y)Z$ is given by

$$R'(X_\varrho, Y_\varrho)Z_\varrho = (R(X,Y)Z)_\varrho + \underset{r}{*}\partial_\circ.$$

Now, using formula (d), we get

$$D'^2 X_\varrho/dt^2 = (D^2 X/dt^2)_\varrho + **_r\, \partial_\circ,$$

where

$$**_r = -^\circ L(D^2 X/dt^2).$$

Therefore, using formula (a), we get

$$D'^2 X_\varrho/dt^2 + R'(V_\varrho, X_\varrho)V_\varrho = (D^2 X/dt^2 + R(V,X)V)_\varrho + r\, \partial_\circ,$$

where

$$r = *_r + **_r, \quad V = dc/dt \quad \text{and} \quad V_\varrho \text{ is its canonical lift.}$$

Thus, we have

Theorem 2. *If the 1-form $b_i dx^i$ is closed, then the lifting of a Jacobi field J along a geodesic c in M_n is a Jacobi field along the canonical lift c_ϱ in M_{n+1}.*

A point p of a geodesic c, corresponding to the parameter value $t = b$, is said to be cojugate [6] to the point q, corresponding to the parameter value $t = a$, $a < b$, along c if there exists a non zero Jacobi field J along c which vanishes for $t = a$ and $t = b$.

The multiplicity of the point p as a conjugate point to q is equal to the dimension of the vector space consisting of all such Jacobi fields.

Proposition 2. *If the 1-form $b_i dx^i$ is closed, then the lifting of a conjugate point of c will be a conjugate point of c_ϱ.*

Proof. If the 1-form $b_i dx^i$ is closed, then, using theorem 2, the lift of a Jacobi field along a geodesic c in M_n is a Jacobi field along the canonical lift c_ϱ in M_{n+1}. By the definition of conjugate points, the result follows.

Therefore, we have

Theorem 3. *If the 1-form $b_i dx^i$ is closed, then every geodesic c of $(M_n, {}^\circ L)$ and its canonical lift c_ϱ of $(M_{n+1}, \mathcal{L})$ have the same index.*

§5. The projecting.

For any point z of M_{n+1}. the tangent space of M_{n+1} at z is written as the direct sum $T_Z M_{n+1} = T_Z R \oplus T_Z W_n$. Therefore, any tangent vector Z to M_{n+1} at z is written in a unique manner as $Z = Z_\circ + Z_p$, where $Z_\circ = Z^\circ \partial_\circ$ and $Z_p = Z^i \partial_i$.

Let X be any vector field on M_{n+1}, then X can be written uniquely in the form:

$$X = X^i \partial_i + X^\circ \partial_\circ.$$

In general X^i and X° are functions of x^i and x°. If X^idepends on x^i alone (i.e.

X^i is independent of x°), then, clesrly $X^i\partial_i$ is a vector field on M_n. In this case, we say that X is projectable and its projection on M_n is $X_p = X^i\partial_i$. In the following, unless otherwise stated, all vector fields on M_{n+1} which we will deal with are assumed to be projectable vector fields.

We shall denote by $X, Y, Z, \ldots$ the vector fields on M_{n+1}, and their projections on M_n by $X_p, Y_p, Z_p, \ldots$.

It is clear that: the projection of the sum of two vector filds is equal to the sum of their projections.

The bracket two vector fields on M_{n+1} can be decomposed into

$$[X,Y] = [X,Y]_\circ + [X,Y]_p \tag{21}$$

where

$$[X,Y]_p = X_pY_p - Y_pX_p = [X_p, Y_p].$$

That is; the projection of the bracket is equal to the bracket of the projections.

Let ∇' (respectively ∇) be the Riemannian connection on M_{n+1} (respectively $(M_n, {}^\circ L)$).

The covariant derivative of a vector field Y in the direction of a vector X can be decomposed into

$$(\nabla'_X Y)_p = \nabla_{X_p} Y_p + {}^*C(X,Y) \tag{22}$$

where

$${}^*C(X,Y) = (q^k_{\circ j} X^\circ Y^j + q^k_{i\circ} X^i Y^\circ + \mathring{A}^k_{ij} X^i Y^j)\partial_k.$$

It is clear that:

1) ${}^*C(X,Y) = {}^*C(Y,X)$,

2) if the 1-form $b_i dx^i$ is closed, then (22) reduces to

$$(\nabla'_X Y)_p = \nabla_{X_p} Y_p. \tag{23}$$

Let R' (respectively R) be the curvature tensor of the connection ∇' (respectively ∇). Thus, we have

$$(R'(X,Y)Z)_p = R(X_p, Y_p)Z_p + {}^*\tilde{A}(X,Y)Z \tag{24}$$

wher

$$\begin{aligned}{}^*\tilde{A}(X,Y)Z =&[R'^h_{ijo}X^iY^jZ^o + R'^h_{ioo}X^iY^oZ^o \\ &+ R'^h_{iok}X^iY^oZ^k + R'^h_{ojo}X^oY^jZ^o \\ &+ R'^h_{ojk}X^oY^jZ^k + \{(\partial_j A^h_{ik} + q^o_{ik}q^h_{jo}) \\ &+ \mathring{A}^r_{ik}\{{h \atop jr}\} + q^r_{ik}\mathring{A}^h_{jr})X^iY^jZ^k, \\ &- (i/j)]\partial_h,\end{aligned}$$

and (means the same terms repeated with i and j interchanged.

Remark. We have to point out that if the 1-form $b_i dx^i$ is closed, then ${}^*\tilde{A}(X,Y)Z = 0$, and therefore the projection of a flat space will be a flat space.

Let c_ϱ be the canonical lift in M_{n+1} of the geodesic c in M_n. Let D'/dt and D/dt be the covariant derivative operator associated with the Riemannian connections ∇' on M_{n+1} and ∇ on M_n respectively. If X is a vector field along c_ϱ, then its covariant derivative along c_ϱ is $D'X/dt$ and its projection is given by

$$(D'X/dt)_p \;=\; (dX^h/dt + q^h_{\alpha\beta}\dot{x}^\alpha X^\beta)\partial_h.$$

Therefore, by using (23), we have

Lemma 3. *If the 1-form $b_i dx^i$ is closed, then the covariant derivative associated with the connection ∇ of the projection X_p along c of a vector field X along c_ϱ is equal to the projection of the covariant derivative associated with the connection ∇' of the vector field X.*
That is to say;

$$(D'X/dt)_p \;=\; DX_p/dt. \tag{25}$$

From this we can see that:

If the 1-form $b_i dx^i$ is closed, then the projection of a parallel vector field along c_ϱ with respect to ∇' is also a parallel vector field along c with respect to ∇.

From equations (24), (25) and the precedent remark, we have: If the 1-form $b_i dx^i$ is closed, then

$$(D'^2X/dt^2 + R'(V,X)V)_p \;=\; D^2X_p/dt^2 + R(V_p,X_p)V_p, \tag{26}$$

where

$$V = dc_\varrho/dt \quad \text{is the velocity vector field along} \quad c_\varrho.$$

Thus, we have

Theorem 4. *If the 1-form $b_i dx^i$ is closed, then the projection of a Jacobi field along the canonical lift c_ϱ will be a Jacobi field along c.*

Using theorem 4, a similar result to theorem 3, for projection of geodesics, can be obtained.

§6. Examples.

Example 1. Let $L = \sqrt{(\dot{x}^2 + \dot{y}^2)} + y\dot{x} + x\dot{y}$ be a Lagrangian function on $M_n = R^2$.
Here, the scalar 1-form $b_i dx^i$ is exact and is equal to $d(xy)$. The extremals of the given Randers space are straight lines, while the extremals of the Riemannian space with lagrangian $\mathcal{L} = \sqrt{\dot{x}^2 + \dot{y}^2 + (\dot{z} + y\dot{x} + x\dot{y})^2}$ are parabolas which be projected as straigt lines on the (x, y) plane.
For, it s° is the arc length defined by ${}^\circ L$ (the Euclidean space), then the extremals of L are given by

$$\begin{cases} x = x_\circ + s^\circ \cos\phi, \\ y = y_\circ + s^\circ \sin\phi. \end{cases} \tag{1}$$

To determine $z = x^\circ$, we integrate $dz + xdy + ydx = ds^\circ$ which, with (1),gives

$$z = z_\circ - (s^{\circ 2}/2)\sin 2\phi - s^\circ(x_\circ \sin\phi + y_\circ \cos\phi - 1).$$

Thus the extremals of $\mathcal{L}$ satisfying $\partial\mathcal{L}/\partial\dot{z} = h$ are parabolas in vertical planes which have vertical axis.

Example 2. Let $L = \sqrt{(\dot{x}^2 + \dot{y}^2)} + y\dot{x} - x\dot{y}$ be a lagrangian function on $M_n = R^2$.

The extremals of the given Randers space are circles with radii 1/2.

For simplicity we take s° as a parameter such that $ds^{\circ 2} = dx^2 + dy^2$. Then, the Euler-Lagrange equations are

$$\begin{aligned} d(dx/ds^\circ + y)/ds^\circ + dy/ds^\circ &= 0, \\ d(dy/ds^\circ - x)/ds^\circ - dx/ds^\circ &= 0. \end{aligned}$$

By integration, we get

$$\begin{cases} x = x_\circ + (1/2)\cos(2s^\circ + \phi), \\ y = y_\circ + (1/2)\sin(2s^\circ + \phi). \end{cases} \tag{1}$$

The extremals of $\mathcal{L} = \sqrt{{}^\circ L^2 + \beta^2}$ on R^3 are given by (1) and

$$dz/ds^\circ + ydx/ds^\circ - xdy/ds^\circ = 1,$$

where x and y are given by equation (1) and $z = x^\circ$.
Therefore,

$$ydx/ds^\circ - xdy/ds^\circ = -(1/2 + x_\circ \cos(2s^\circ + \phi) + y_\circ \sin(2s^\circ + \phi))$$

which gives

$$dz/ds^\circ = 3/2 + x_\circ(2s^\circ + \phi) + y_\circ \sin(2s^\circ + \phi).$$

By integration, we get

$$z = z_\circ + \frac{1}{2}x_\circ \sin(2s^\circ + \phi) - \frac{1}{2}y_\circ \cos(2s^\circ + \phi) + (\frac{3}{2})s^\circ \tag{2}$$

Equations (1) and (2) define the extremals of $\mathcal{L}$.
As a special case, if the circle's center is the origin, i.e. $x_\circ = y_\circ = 0$, then

$$z = z_\circ + (3/2)s^\circ.$$

The extremals of $\mathcal{L}$ are, in this case, helices.

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