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ON A CLASS OF ANALYTIC FUNCTIONS INVOLVING THE SALAGEAN DIFFERENTIAL OPERATOR

S. ABDUL HALIM

Abstract. We introduce the class $B_n(\alpha)$ consisting of functions analytic in the unit disc. In this paper, we give some properties of this class as well as considering some integral operators.

1. Introduction

Let A be the calss of functions f analytic in the unit disc $D = \{z : |z| < 1\}$ and of the form

$$f(z) \ = \ z + \sum_{j=2}^{\infty} a_j z^j. \tag{1.1}$$

For a function $f \in A$, we consider the differential operator D^n introduced by Salagean [4].

Definition 1.1. Let $f \in A$. For $n \geq 1$, define D^n by

$$\begin{aligned} D^1 f(z) &= Df(z) \ = \ zf'(z), \\ D^n f(z) &= D(D^{n-1} f(z)) \ = \ z\Big[D^{n-1} f(z)\Big]', \end{aligned} \tag{1.2}$$

where $D^o f(z) = f(z)$.

With the above definition, we now introduce the subclass $B_n(\alpha)$ of A as follows:–

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Defintion 1.2. For $\alpha > 0$ and $n = 0, 1, 2, \ldots$, a function f normalized by (1.1) is said to be in $B_n(\alpha)$, if, and only if,

$$\operatorname{Re} \frac{D^n[f(z)^\alpha]}{z^\alpha} > 0, \tag{1.3}$$

for $z \in D$. (Powers in (1.3) are meant as principal values).

In the case $n = 1$, $B_1(\alpha)$ denotes the class of Bazilevic functions with logarithmic growth. (see [5] and [6]). Earlier, in [2] it was proven that $B_1(\alpha) \subset S$, where S is the subclass of A, consisting of univalent functions in D. Also let P denote the class of analytic functions p such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $Z \in D$. In this paper, we give some properties of the class $B_n(\alpha)$ and consider an iterated integral operator problem.

2. Preliminary Lemmas.

We shall need the following lemmas.

Lemma 2.1. [3]. *Let M and N be analytic in D with $M(0) = N(0) = 0$. If $N(z)$ maps D onto a many sheeted region which is starlike with respect to the origin and $\operatorname{Re} \frac{M'(z)}{N'(z)} > 0$ in D, then $\operatorname{Re} \frac{M(z)}{N(z)} > 0$ in D.*

Lemma 2.2. *Let $p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i \in P$. Then for $\alpha + c > 0$,*

$$q(z) = 1 + (\alpha + c) \sum_{i=1}^{\infty} \frac{c_i z^i}{(\alpha + c + i)}, \quad z \in D$$

in also in P.

Proof. Since $p \in P$, we then have

$$\operatorname{Re} \left[\frac{z^{\alpha+c-1} + \sum_{i=1}^{\infty} c_i z^{i+\alpha+c-1}}{z^{\alpha+c-1}} \right] > 0.$$

Thus

$$\operatorname{Re} \left[\frac{\left[\frac{z^{\alpha+c}}{\alpha + c} + \sum_{i=1}^{\infty} \frac{c_i z^{i+\alpha+c}}{\alpha + c + i} \right]'}{\left[\frac{z^{\alpha+c}}{\alpha + c} \right]'} \right] > 0$$

which by Lemma 2.1 above, gives our required result.

3. Some properties of $B_n(\alpha)$.

Theorem 3.1. *$B_{n+1}(\alpha) \subset B_n(\alpha)$ for $n \geq 1$.*

Proof. Let $f \in B_{n+1}(\alpha)$, then by (1.3), we have

$$\mathrm{Re}\,\frac{D^{n+1}f(z)^\alpha}{z^\alpha} > 0,$$

and so (1.2) gives

$$\mathrm{Re}\,\frac{\alpha\left[D^n f(z)^\alpha\right]'}{(z^\alpha)'} > 0,$$

which by Lemma 2.1 above, implies that

$$\mathrm{Re}\,\frac{D^n f(z)^\alpha}{z^\alpha} > 0.$$

Hence $f \in B_n(\alpha)$.

Corollary. *$B_n(\alpha) \subset S$ for $n = 1, 2, 3, \ldots$.*

Proof. This is easily seen, since $B_n(\alpha) \subset B_1(\alpha)$ for $n = 2, 3, 4, \ldots$ and $B_1(\alpha) \subset S$.

Theorem 3.2. *Let $f \in B_n(\alpha)$ and $\alpha + c > 0$. Then the function F defined by*

$$F(z)^\alpha = \frac{\alpha + c}{z^c}\int_0^z t^{c-1} f(t)^\alpha dt, \quad (z \in D) \tag{3.1}$$

is also in the class $B_n(\alpha)$.

Proof. Since $f \in B_n(\alpha)$, there exists $p \in P$ with $p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$ such that

$$\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n p(z). \tag{3.2}$$

(1.2) and (3.2) give

$$\frac{D^{n-1} f(z)^\alpha}{z^\alpha} = \alpha^{n-1}\left[1 + \alpha \sum_{j=1}^{\infty} \frac{c_j z^j}{(j + \alpha)}\right].$$

Now, on assumption that

$$\frac{D^{n-k}f(z)^\alpha}{z^\alpha} = \alpha^{n-k}\left[1+\alpha^k\sum_{j=1}^{\infty}\frac{c_j z^j}{(j+\alpha)^k}\right]$$

is true for some $0 < k \le n$, we have

$$\left[\frac{f(z)}{z}\right]^\alpha = 1+\alpha^n\sum_{j=1}^{\infty}\frac{c_j z^j}{(j+\alpha)^n}, \tag{3.3}$$

which follows trivally by induction.

Using (3.3) in (3.1) gives

$$\left[\frac{F(z)}{z}\right]^\alpha = 1+(\alpha+c)\sum_{j=1}^{\infty}\frac{\alpha^n c_j z^j}{(j+\alpha)^n(j+\alpha+c)}.$$

Next, assume that

$$\alpha^{-k}\frac{D^k F(z)^\alpha}{z^\alpha} = 1+\alpha^{n-k}(\alpha+c)\sum_{j=1}^{\infty}\frac{c_j z^j}{(j+\alpha)^{n-k}(j+\alpha+c)}$$

is true for $0 < k \le n$.

Hence

$$\alpha^{-n}\frac{D^n F(z)^\alpha}{z^\alpha} = 1+(\alpha+c)\sum_{j=1}^{\infty}\frac{c_j z^j}{(j+\alpha+c)},$$

follows by induction.

Finally, since $p \in P$, by Lemma 2.2 we have

$$\operatorname{Re}\frac{D^n F(z)^\alpha}{z^\alpha} > 0,$$

i.e. $F \in B_n(\alpha)$.

Theorem 3.3. *Let $f \in B_n(\alpha)$. Then for $n \ge 0$ and $z = re^{i\theta} \in D$,*

$$1+2\alpha^n\sum_{j=1}^{\infty}\frac{(-r)^j}{(j+\alpha)^n} \le \operatorname{Re}\left[\frac{f(z)}{z}\right]^\alpha \le \left|\frac{f(z)}{z}\right|^\alpha \le 1+2\alpha^n\sum_{j=1}^{\infty}\frac{r^j}{(j+\alpha)^n}.$$

The result is sharp.

Proof.

(i) We first prove the lower bound.

From (3.2), since $p \in P$, we have

$$\alpha^{-n}\,\mathrm{Re}\,\frac{D^n f(z)^\alpha}{z^\alpha} = \mathrm{Re}\,p(z) \geq \frac{1-r}{1+r} = 1+2\sum_{j=1}^{\infty}(-r)^j.$$

Next, for some $0 < k \leq n$, assume that

$$\alpha^{-k}\,\mathrm{Re}\,\frac{D^k f(z)^\alpha}{z^\alpha} \geq 1+2\alpha^{n-k}\sum_{j=1}^{\infty}\frac{(-r)^j}{(j+\alpha)^{n-k}} \tag{3.4}$$

Then (1.2) gives

$$\begin{aligned}\alpha^{1-k}\mathrm{Re}\frac{D^{k-1} f(z)^\alpha}{z^\alpha} &= \alpha^{1-k}\mathrm{Re}\frac{1}{z^\alpha}\int_0^z \left[D^{k-1} f(t)^\alpha\right]' dt\\ &= \frac{\alpha^{1-k}}{r^\alpha}\int_0^r \rho^{\alpha-1}\mathrm{Re}\frac{D^k f(\rho e^{i\theta})^\alpha}{(\rho e)^{i\theta})^\alpha} d\rho, \text{ where } t=\rho e^{i\theta},\\ &\geq \frac{\alpha^{1-k}}{r^\alpha}\int_0^r \rho^{\alpha-1}\left[\alpha^k + 2\alpha^n\sum_{j=1}^{\infty}\frac{(-\rho)^j}{(j+\alpha)^{n-k}}d\rho,\right.\end{aligned}$$

on using (3.4).

Hence the lower bound follows by induction.

(ii) We next prove the upper bound.

Again, we use induction. Obviously, due to $|p(z)| \leq \frac{1+r}{1-r}$, we have from (3.2),

$$\alpha^{-n}\left|\frac{D^n f(z)^\alpha}{z^\alpha}\right| \leq \frac{1+r}{1-r} = 1+2\sum_{j=1}^{\infty} r^j.$$

For fixed n, assume that when $0 < k \leq n$,

$$\alpha^{-k}\left|\frac{D^k f(z)^\alpha}{z^\alpha}\right| \leq 1+2\alpha^{n-k}\sum_{j=1}^{\infty}\frac{r^j}{(j+\alpha)^{n-k}}$$

Then

$$\alpha^{1-k} \mid \frac{D^{k-1}f(z)^\alpha}{z^\alpha} \mid \leq \frac{\alpha^{1-k}}{r^\alpha}\int_0^r \rho^{\alpha-1} \mid \frac{D^k f(\rho e^{i\theta})^\alpha}{(\rho e^{i\theta})^\alpha} \mid d\rho$$
$$\leq 1 + 2\alpha^{n-(k-1)}\sum_{j=1}^\infty \frac{r^j}{(j+\alpha)^{n-k+1}}.$$

The proof now follows by induction. For fixed n, the result is sharp, since the upper and lower bounds for $\operatorname{Re} p(z)$ and $|p(z)|$ are sharp.

4. Iterated integral operator.

In [1], the authors gave the sharp estimate for the lower bound of $\operatorname{Re}(f(z)/z)^\alpha$ when $f \in B_1(\alpha)$. This estimate can be seen in Theorem 3.3 in the case $n = 1$. This result was then extended to include iterated integrals.

We now will give the sharp result for a more generalized version of iterated integral operator. First, for $z \in D$, $a > -1$ and $m = 1, 2, 3, \ldots$ let

$$I_m(z) = \frac{a+1}{z^{a+1}}\int_0^z t^a\, I_{m-1}(t)dt, \tag{4.1}$$

where $I_0(z) = (f(z)/z)^\alpha$.

Theorem 4.1. *For a fixed n, let $f \in B_n(\alpha)$. Then for $z = re^{i\theta} \in D$ and $m \geq 0$,*

$$\operatorname{Re} I_m(z) \geq \gamma_m(r)$$

and

$$\gamma_m(r) < 1,$$

where for $m = 0, 1, 2, \ldots$

$$\gamma_m(r) = 1 + 2\alpha^n(1+a)^m\sum_{j=1}^\infty \frac{(-r)^j}{(j+\alpha)^n(j+a+1)^m}. \tag{4.2}$$

Equality is attained for f where $\left[\frac{D^n f(z)^\alpha}{z^\alpha}\right] = \alpha^n\left[\frac{1-z}{1+z}\right]$.

Proof. From Theorem 3.3, we have

$$\mathrm{Re}\, I_0(z) \;\geq\; 1 + 2\alpha^n \sum_{j=1}^{\infty} \frac{(-r)^j}{(j+\alpha)^n}.$$

Next, from (4.1)

$$\begin{aligned}
\mathrm{Re}\, I_{k+1}(z) &= \mathrm{Re}\, \frac{a+1}{z^{a+1}} \int_0^z t^a\, I_k(t)dt \\
&= \frac{a+1}{r^{a+1}} \int_0^r \rho^a \mathrm{Re}\, I_k(\rho e^{i\theta}) d\rho \\
&\geq \frac{a+1}{r^{a+1}} \int_0^r \rho^a \Big[1 + 2\alpha^n (1+a)^k \sum_{j=1}^{\infty} \frac{(-\rho)^j}{(j+\alpha)^n (j+a+1)^k}\Big] d\rho \\
&= 1 + 2\alpha^n (1+a)^{k+1} \sum_{j=1}^{\infty} \frac{(-r)^j}{(j+\alpha)^n (j+a+1)^{k+1}} \\
&= \gamma_{k+1}(r).
\end{aligned}$$

Hence, the first inequality follows by induction.

Now, for $m \geq 0$, the series

$$\gamma_m(r) \;=\; 1 + 2\alpha^n (1+a)^m \sum_{j=1}^{\infty} \frac{(-r)^j}{(j+\alpha)^n (j+a+1)^m}$$

is absolutely convergent. On rearranging pairs of terms in $\gamma_m(r)$ as

$$\begin{aligned}
&\gamma_m(r) \\
&= 1 - 2\alpha^n (1+a)^m \Big[\frac{r}{(1+\alpha)^n (2+a)^m} - \frac{r^2}{(2+\alpha)^n (3+a)^m}\Big] \\
&\quad - 2\alpha^n (1+a)^m \Big[\frac{r^3}{(3+\alpha)^n (4+a)^m} - \frac{r^4}{(4+\alpha)^n (5+a)^m}\Big] + \dots \\
&= 1 - 2\alpha^n (1+a)^m \sum_{j=1}^{\infty} \Big[\frac{r^{2k-1}}{(2k-1+\alpha)^n (2k+a)^m} - \frac{r^{2k}}{(2k+\alpha)^n (2k+1+a)^m}\Big]
\end{aligned}$$

shows that $\gamma_m(r) < 1$.

This completes the proof.

Remark. Finally, we conjecture that for $\gamma_m(r)$ given by (4.2), it also satisfies $\gamma_m(r) > \gamma_m(1)$.

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References

[1] S. Abdul Halim and D. K. Thomas, "A Note on Bazilevic Functions", *Internat. J. Math. and Math. Sci. To appear*

[2] I. E. Bazilevic, "On a case of integrability in quadratures of the Loewner - Kufarev equation", *Math. Sb. N. S.* 37 (1955), pp.471-476.

[3] R. J. Libera, "Some classes of regular Univalent functions", *Proc. Amer. Math. Soc.* 16 (1965), pp. 755-758.

[4] G. S. Salagean, "Subclasses of univalent functions", *Lecture notes in Math* 1013 (1983), pp. 362-372.

[5] R. Singh, "On Bazilevic functions", *Proc. Amer. Math. Soc.* 38(2) (1973), pp. 261-271.

[6] D. K. Thomas, "On a subclass of Bazilevic functions", *Internat. J. Math. and Math. sci.* 8(4) (1985), pp. 779-783.

Department of Mathematics, University of Malaya, 59100, Kuala Lumpur, Malaysia.