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ON THE SOLUTION SETS OF DIFFERENTIAL INCLUSION IN BANACH SPACES

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Abstract. We prove the set of all classical solutions of the differential inclusion

$$\dot{x}(t) \in A(t,x) + F(t,x,\dot{x})$$
$$x(t_0) = x_0, \quad \dot{x}(t_0) = y_0$$

is a retract of the space C^1.

1. Introduction.

Let $I \subset R$ be a compact interval; E a Banach space, F a multifunction from $I \times E \times E$ into the subsets of E; A a continuous functions from $I \times E$ into E.

Given $t_0 \in I$, $x_0 \in E$, $y_0 \in A(t_0, x_0) + F(t_0, x_0, y_0)$ consider the problem

$$\begin{aligned} \dot{x} &\in A(t,x) + F(t,x,\dot{x}) \\ x(t_0) &= x_0 \\ \dot{x}(t_0) &= y_0 \end{aligned} \tag{1}$$

A functions $\phi : I \to E$ is said to be a classical solution of (1) if $\phi \in C^1$ and $\dot{\phi}(t) \in A(t, \phi(t)) + F(t, \phi(t), \dot{\phi}(t))$ for all $t \in I$, $\phi(t_0) = x_0$, $\dot{\phi}(t_0) = y_0$.

In this paper we prove that under suitable assumptions the set of all classical solutions of (1), $\Gamma(t_0, x_0, y_0, F)$ is a retract of C^1 which depends in a Lipschitzian

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way on x_0, y_0, F. This is also true for the solution set of $\dot{x} \in A(t,x) + F(t,x,\dot{x})$,. $x(t_0) = x_0$. In [7] Ricceri discussed the problem without the term $A(t,x)$. Here our assumptions do not imply the compactness of $\Gamma(t_0, x_0, y_0, F)$ not even in C. Our result is different from many other recent works [1-3,5].

2. Preliminaries and Notations.

Let X, Y be two nonempty sets. A multifunction Φ from X into Y ($\Phi : X \to 2^Y$) is a function from X into the family of all nonempty subsets of Y. When X, Y are two topological spaces, we say that Φ is lower (upper) semicontinous if for every open (closed) set $\Omega \subset Y$, the set $\{x \in X : \Phi(x) \cap \Omega \neq \phi\}$ is open (closed) in X. A single valued function $f : X \to Y$ is said to be a selection of Φ if $f(x) \in \Phi(x)$ for all $x \in X$.

If (Σ, δ) is a metric space, for every $x \in \Sigma$ and nonempty $A, B \subset \Sigma$ we put

$$\delta(x, A) = \inf_{z \in A} \delta(x, z); \ \delta^*(A, B) = \sup_{z \in A} \delta(z, B)$$

$$\delta_H(A, B) = \max\{\delta^*(A, B), \delta^*(B, A)\}.$$

Let (X, d), (Y, ρ) be two metric spaces. A multifunction $\Phi : X \to 2^Y$ is said to be Lipschitzian if there exists a real number $L \geq 0$ (Lipschitz constant) such that

$$\rho_H(\Phi(x), \Phi(z)) \leq Ld(x, z)$$

for all $x, z \in X$. If $L < 1$ we say that Φ is a multivalued contraction. Observe that any Lipschitzian multifunction is lower semicontinuous.

If I is a compact real interval and $(E, ||\cdot||)$ is a real Banach space, we denote by C the space of all continuous functions from I into E, equipped with the norm

$$|| \phi ||_C = \max_{t \in I} || \phi(t) ||$$

We dentoe by C^1 the space of all continuously (strongly) differentiable functions from I into E, equipped with the norm

$$|| \phi ||_{C^1} = || \phi ||_C + || \dot{\phi} ||_C$$

where $\dot{\phi}$ is the (strong) derivative of ϕ.

A set A in a topological space X is said to be a retract of X if $A \neq \emptyset$ and there exists a continuous function $P : X \to A$ such that $P(x) = x$ for all $x \in A$.

3. Main Result

Now $(E, \|\cdot\|)$ is a real Banach space; d is the metric induced by $\|\cdot\|$. I is a (non-degenerate) compact real interval. Let $\alpha, \beta, \gamma : I \to [0, \infty)$ be continuous functions such that $\beta(t) < 1$, $\forall t \in I$. Denote $\mathcal{L}_{\alpha,\beta,E}$ be the family of all lower semicontinuous multifunctions $F : I \times E \times E \to 2^E$, with closed and convex values such that for every $t \in I$, $x, y, u, v \in E$ one has

$$d_H(F(t,x,y), F(t,u,v)) \leq \alpha(t) \,||\, x - u \,||\, + \beta(t) \,||\, y - v \,||$$

Let $A : I \times E \to E$ be a continuous function satisfying for every $x, y \in E$, $t \in I$,

$$d(A(t,x), A(t,y)) \leq \gamma(t) \,||\, x - y \,||$$

Let $t_0 \in I$, $x_0 \in E$, $F : I \times E \times E \to 2^E$, $A : I \times E \to E$. Then put

$$\Gamma(t_0, x_0, F) = \{\phi \in C^1 : \phi(t) \in A(t, \phi(t)) + F(t, \phi(t), \dot{\phi}(t)) \text{ for all } t \in I, \phi(t_0) = x_0\}$$

Further if $y_0 \in A(t_0, x_0) + F(t_0, x_0, y_0)$ we put

$$\Gamma(t_0, x_0, y_0, F) = \{\phi \in \Gamma(t_0, x_0, F) : \dot{\phi}(t_0) = y_0\}.$$

Before stating our theorem, we recall a lemma which will be applied in the theorem. It follows at once from [1] and [2].

Lemma. *Let X be a paracompact topological space; $(Y, \|\cdot\|_Y)$ a Banach space; $\Phi : X \to 2^Y$ a lower semicontinuous multifunction with closed and convex values; $f : X \to Y$ a continuous function; $\beta : X \to [0, \infty)$ a lower semicontinuous function such that $\delta(f(x), \Phi(x)) \leq \beta(x)$ for all $x \in X$, where δ is the metric* *induced by* $\|\cdot\|_Y$. *Then for every* $\varepsilon > 0$, *there exists a continuous selection* g *of* Φ *such that* $\|g(x) - f(x)\|_Y \leq \beta(x) + \varepsilon$ *for all* $x \in X$.

Theorem. *Let* $F \in \mathcal{L}_{\alpha,\beta,E}$. *Then for every* $t_0 \in I$, $x_0 \in E$, $y_0 \in A(t_0, x_0) + F(t_0, x_0, y_0)$, *each of the sets* $\Gamma(t_0, x_0, F)$, $\Gamma(t_0, x_0, y_0, F)$ *is a retract of the space* C^1. *Further if* F *is a singlevalued function, then the set* $\Gamma(t_0, x_0, F)$ *is singleton.*

Proof. Fix $t_0 \in I$, $x_0 \in E$, $y_0 \in A(t_0, x_0) + F(t_0, x_0, y_0)$ and $L \in (\frac{1}{1-\beta^*}, \infty)$ with $\max_{t\in I} \beta(t) = \beta^* \in [0, 1)$.

For each $\phi \in C$ put

$$|| \phi ||_0 = \max_{t\in I} e^{-L[\gamma(t)+\alpha(t)]|t-t_0|} || \phi(t) ||$$

Hence $\|\cdot\|_0$ is norm equivalent to $\|\cdot\|_C$.

Next, for every $\Psi \in C$ put

$$\begin{aligned}\Phi(x_0, \Psi, F) &= \{\phi \in C;\ \phi(t) \in A(t, x_0 + \int_{t_0}^{t} \Psi(\tau)d\tau) \\ &\quad + F(t, x_0 + \int_{t_0}^{t} \Psi(\tau)d\tau, \Psi(t)) \qquad \text{for all } t \in I\}\end{aligned}$$

Here $\int_{t_0}^{t} \Psi(\tau)d\tau$ is Riemann integral.

Put $C_{y_0} = \{\phi \in C;\ \phi(t_0) = y_0\}$ as well as

$$\Phi(x_0, y_o, \Psi, F) = \Phi(x_0, \Psi, F) \cap C_{y_0} \quad \text{for each} \quad \Psi \in C_{y_0}$$

From [4], it follows that $\Phi(x_0, \Psi, F) \neq \emptyset$ for all $\Psi \in C$ and that $\Phi(x_0, y_0, \Psi, F) \neq \emptyset$ for all $\Psi \in C_{y_0}$.

We now claim that the multifunctions $\Phi(x_0, \cdot, F)$ and $\Phi(x_0, y_0, \cdot, F)$ are multi-valued contractions with respect to the metric, say ρ induced by $\|\cdot\|_0$ with Lipschitz constant $\frac{1}{L} + \beta^*$.

We prove this only for $\Phi(x_0, y_0, \cdot, F)$ since the proof for $\Phi(x_0, \cdot, F)$ is completely similar.

Fix $\Psi, \omega \in C_{y_0}$, $\phi \in \Phi(x_0, y_0, \Psi, F)$. Then for every $t \in I$, we have

$$d(\phi(t), A(t, x_0 + \int_{t_0}^{t} \omega(\tau)d\tau) + F(t, x_0 + \int_{t}^{t_0} \omega(\tau)d\tau, \omega(t)))$$

$$\leq \gamma(t) \| \int_{t_0}^{t} (\Psi(\tau) - \omega(\tau))d\tau \| + \alpha(t) \| \int_{t_0}^{t} (\Psi(\tau) - \omega(\tau))d\tau \| + \beta(t) \| \Psi(t) - \omega(t) \|$$

Define a multifunction $H : I \to 2^E$ by

$$H(t) = \begin{cases} A(t, x_0 + \int_{t_0}^{t} \omega(\tau)d\tau) + F(t, x_0 + \int_{t_0}^{t} \omega(\tau)d\tau, \omega(t)) & \text{if } t \neq t_0, \ t \in I \\ \{y_0\} & \text{if } t = t_0 \end{cases}$$

Then clearly H is lower semicontinuous [4]. By Lemma, for every $\varepsilon > 0$, H admits a continuous selection h such that

$$\| h(t) - \phi(t) \|$$
$$\leq \gamma(t) \| \int_{t_0}^{t} (\Psi(\tau) - \omega(\tau))d\tau \| + \alpha(t) \| \int_{t_0}^{t} (\Psi(\tau) - \omega(\tau))d\tau \| + \beta(t) \| \Psi(\tau) - \omega(\tau) \| + \varepsilon$$
$$\leq [\gamma(t) + \alpha(t)] \| \int_{t_0}^{t} \Psi(\tau) - \omega(\tau))d\tau \| + \beta(t) \| \Psi(\tau) - \omega(\tau) \| + \varepsilon \qquad (2)$$

for every $t \in I$. In particular observe that $h \in \Phi(x_0, y_0, \omega, F)$.

Now we evaluate $\|h - \phi\|_0$. Assume $a < t_0$ and for every $t \in [a, t_0]$, we have

$$e^{-L[\gamma(t)+\alpha(t)](t_0-t)} \| \int_{t_0}^{t} (\Psi(\tau) - \omega(\tau))d\tau \|$$
$$= \| \int_{t_0}^{t} e^{-L[\gamma(t)+\alpha(t)](\tau-t)} \cdot e^{-L[\gamma(t)+\alpha(t)](t_0-\tau)} (\Psi(\tau) - \omega(\tau))d\tau \|$$
$$\leq \| \Psi - \omega \|_0 \int_{t}^{t_0} e^{-L[\gamma(t)+\alpha(t)](\tau-t)} d\tau$$
$$\leq \frac{1}{L[\gamma(t) + \alpha(t)]} \| \Psi - \omega \|_0$$

Therefore from (2) we have

$$e^{-L[\gamma(t)+\alpha(t)](t_0-t)} \| h(t) - \phi(t) \| \leq [\frac{1}{L} + \beta(t)] \| \Psi - \omega \|_0 + \varepsilon$$
$$\leq (\frac{1}{L} + \beta^*) \| \Psi - \omega \|_0 + \varepsilon$$

Analogously if $t_0 < b$ for every $t \in [t_0, b]$, we obtain

$$e^{-L[\gamma(t)+\alpha(t)](t-t_0)} \| h(t) - \phi(t) \| \leq (\frac{1}{L} + \beta^*) \| \Psi - \omega \|_0 + \varepsilon$$

Hence,

$$\| h - \phi \|_0 \leq (\frac{1}{L} + \beta^*) \| \Psi - \omega \|_0 + \varepsilon$$

Therefore

$$\rho^*(\Phi(x_0, y_0, \Psi, F), \Phi(x_0, y_0, \omega, F)) \leq (\frac{1}{L} + \beta^*) \| \Psi - \omega \|_0$$

Interchanging the roles of Ψ and ω we get

$$\rho^*(\Phi(x_0, y_0, \omega, F), \Phi(x_0, y_0, \Psi, F)) \leq (\frac{1}{L} + \beta^*) \| \Psi - \omega \|_0$$

Since $\frac{1}{L} + \beta^* < 1$, $\Phi(x_0, y_0, \cdot, F)$ is multivalued contraction. Similarly we can show that $\Phi(x_0, \cdot, F)$ is also multivalued contraction.

Now put

$$P(x_0, F) = \{\phi \in C;\ \phi \in \Phi(x_0, \phi, F)\}$$

as well as

$$P(x_0, y_0, F) = \{\phi \in C_{y_0};\ \phi \in \Phi(x_0, y_0, \phi, F)\}$$

Then taking into account that all the sets $\Phi(x_0, \Psi, F)$, $\Phi(x_0, y_0, \Psi, F)$ are convex closed, by Theorem 1 of [6] each of the sets $P(x_0, F)$, $P(x_0, y_0, F)$ is a retract of the space C. In particular to see this for $P(x_0, y_0, F)$, take into account that C_{y_0} being closed and convex, is in turn a retract of C.

Next consider the operator $T : C \to C^1$ defined by $T(\Psi)(t) = \int_{t_0}^{t} \Psi(\tau)d\tau$ for every $\Psi \in C$, $t \in I$. Clearly

$$\Gamma(t_0, x_0, F) = \phi^{(x_0)} + T(P(x_0, F)) \tag{3}$$

$$\Gamma(t_0, x_0, y_0, F) = \phi^{(x_0)} + T(P(x_0, y_0; F)) \tag{4}$$

where $\phi^{(x_0)}$ is the constant function on I taking the value x_0.

Let θ_E be the null element of E. Put

$$V_0 = \{\phi \in C^1 : \phi(t_0) = \theta_E\}$$

Clearly the operator T is a linear homeomorphism from C onto V_0. Therefore, each of the sets $T(P(x_0, F))$, $T(P(x_0, y_0, F))$ is a retract of V_0. But V_0, being closed and convex, is a retract of C^1 and hence each of set $T(P(x_0, F))$, $T(P(x_0, y_0, F))$ is a retract of C^1. Hence from (3) and (4) $\Gamma(t_0, x_0, F)$ and $\Gamma(t_0, x_0, y_0, F)$ are retracts of C^1.

Further if F is a single valued funtion, then from the classical contraction mapping principle of Banach-Caccioppoli, the set $\Gamma(t_0, x_0, F)$ is a singleton.

Hence the proof is completed.

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