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A NOTE ON DISCRETE OPIAL'S INEQUALITY

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1.Introduction

Z.Opial [2] has proved in 1960 the following result:

Theorem A. *If f' is a continuous function on $[0,h]$, and if $f(0)=f(h)=0$ and $f(x)>0$ for $x\in(0,h)$, then*

$$\int_0^h \mid f(x)f'(x)\mid dx \le \frac{h}{4}\int_0^h f'(x)^2dx \tag{1}$$

where the constant $h/4$ is the best possible.

In 1967 J.S.W.Wong [4] has proved the following discrete analogue of (1).

Theorem B. *Let u_i be a nondecreasing sequence of nonnegative real numbers, such that $u_0=0$. Then, for $p\ge 1$, we have*

$$\sum_{i=1}^{n}(u_i-u_{i-1})u_i^p \le \frac{(n+1)^p}{p+1}\sum_{i=1}^{n}(u_i-u_{i-1})^{p+1}. \tag{2}$$

Cheng-Ming Lee [1] generalized (2) in the following form:

Theorem C. *Let u_i be a nondecreasing sequence of nonnegative numbers,such that $u_0=0$.*
If $p,q>0, p+q\ge 1$ or $p,q<0$, then

$$\sum_{i=1}^{n}(u_i-u_{i-1})^q u_i^p \le K_n\sum_{i=1}^{n}(u_i-u_{i-1})^{p+q} \tag{3}$$

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where

$$K_0 = \frac{q}{p+q}$$

and

$$K_n = max\big(K_{n-1} + \frac{pn^{p-1}}{p+q}, \frac{q(n+1)^p}{p+q}\big) \qquad (n = 1, 2, \cdots).$$

If $p > 0, q < 0, p+q \leq 1, p+q \neq 0$ *or* $p < 0, q > 0, p+q \geq 1$, *then*

$$\sum_{i=1}^{n} (u_i - u_{i-1})^q u_i^p \geq C_n \sum_{i=1}^{n} (u_i - u_{i-1})^{p+q}, \tag{4}$$

where

$$C_0 = \frac{q}{p+q} \qquad and \qquad C_n = min\big(C_{n-1} + \frac{pn^{p-1}}{p+q}, \frac{q(n+1)^p}{p+q}\big) \qquad (n = 1, 2, \cdots).$$

If $p, q \geq 1$, *then*

$$K_n = \frac{q(n+1)^p}{p+q};$$

If $p \leq 0, q < 0$, *then*

$$K_1 = 1, \qquad and \qquad K_n = 1 + \frac{p}{p+q} \sum_{i=2}^{n} i^{p-1} \qquad (n = 2, 3, \cdots);$$

If $p \geq 0, p+q < 0$, *then*

$$C_1 = 1, \qquad and \qquad C_n = 1 + \frac{p}{p+q} \sum_{i=2}^{n} i^{p-1} \qquad (n = 2, 3, \cdots).$$

Recently, B.G.Pachpatte [3] has proved a new discrete inequality of Opial type involving a function of n independent variables in the following:

Theorem D. *If* z *is a function from* B *to* R *such that* $\nabla_1 \cdots \nabla_n z(x)$ *exist and*

$z(0, x_2, \cdots, x_n) = z(x_1, 0, x_3, \cdots, x_n) = \cdots = z(x_1, \cdots, x_{n-1}, 0) = 0$. *then*

$$\sum_{B} | z(x) || \nabla_1 \cdots \nabla_n z(x) | \leq \left(\sum_{B} (\prod_{i=1}^{n} x_i) (\sum_{B_x} | \nabla_1 \cdots \nabla_n z(y) |^2) \right)^{\frac{1}{2}} \bullet \left(\sum_{B} | \nabla_1 \cdots \nabla_n z(x) |^2 \right)^{\frac{1}{2}}, \tag{5}$$

where

$$N_0 = \{0, 1, 2, \cdots\}, \qquad B = \prod_{i=1}^{n} [0, C_i] \subset N_0^n,$$

and $\nabla_1 z(x_1, x_2, \cdots, x_n) = z(x_1, x_2, \cdots, x_n) - z(x_1 - 1, x_2, \cdots, x_n)$,

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$\nabla_n z(x_1, x_2, \cdots, x_n) = z(x_1, \cdots, x_{n-1}, x_n) - z(x_1, \cdots, x_{n-1}, x_n - 1)$,
$\nabla_1 \nabla_2 z(x_1, x_2, \cdots, x_n) = \nabla_1 [z(x_1, x_2, \cdots, x_n) - z(x_1, x_2 - 1, x_3, \cdots, x_n)]$,

$$\sum_{B} z(x) = \sum_{x_1=1}^{C_1} \cdots \sum_{x_n=1}^{C_n} z(x_1, x_2, \cdots, x_n),$$

$$\sum_{B_x} z(y) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} z(y_1, y_2, \cdots, y_n).$$

The purpose of the present note is to generalize the inequality (2) as well as the inequality (5).

2.Discrete Opial's inequalities in one-dimention

Lemma 1. Let $u_0, u_1, \cdots u_n$ *be a sequence of real numbers,such that* $u_0 = 0$. *If* $p, q \geq 1$, *then*

$$\sum_{i=1}^{n} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(n+1)^p}{p+q} \sum_{i=1}^{n} | u_i - u_{i-1} |^{p+q} \tag{6}$$

Proof. Let $x_i = | u_i - u_{i-1} |$. *Then* $| u_i | \leq \sum_{j=1}^{i} x_j$, so that

$$\sum_{i=1}^{n} | u_i - u_{i-1} |^q | u_i |^p \leq \sum_{i=1}^{n} x_i^q \left(\sum_{j=1}^{i} x_j \right)^p . \tag{7}$$

It suffices to prove

$$\sum_{i=1}^{n} x_i^q \left(\sum_{j=1}^{i} x_j \right)^p \leq \frac{q(n+1)^p}{p+q} \sum_{i=1}^{n} x_i^{p+q} \tag{8}$$

We will prove (8) by induction.

Since $2^p \geq p+1$ for $p \geq 1$, it follows that

$$x_1^{p+q} \leq \frac{q \cdot 2^p}{p+q} x_1^{p+q}.$$

This shows that (8) holds if $n = 1$. Assume (8) holds for $n = k$, and observe

$$\begin{aligned} \sum_{i=1}^{k+1} x_i^q \left(\sum_{j=1}^{i} x_j \right)^p &= \sum_{i=1}^{k} x_i^q \left(\sum_{j=1}^{i} x_j \right)^p + x_{k+1}^q \left(\sum_{j=1}^{k+1} x_j \right)^p \\ &\leq \frac{q(k+1)^p}{p+q} \sum_{i=1}^{k} x_i^{p+q} + x_{k+1}^q \left(\sum_{j=1}^{k+1} x_j \right)^p \\ &= \frac{q(k+1)^p}{p+q} \left(\sum_{i=1}^{k} x_i^{p+q} + \frac{p+q}{q} x_{k+1}^q y_{k+1}^p \right), \end{aligned} \tag{9}$$

where $y_n = \frac{1}{n} \sum_{i=1}^{n} x_i$.

By Young's inequality we have

$$\begin{aligned} \frac{p+q}{q} x_{k+1}^q y_{k+1}^p &\leq \frac{p+q}{q} \left(\frac{q}{p+q} x_{k+1}^{q \frac{p+q}{q}} + \frac{p}{p+q} y_{k+1}^{p \frac{p+q}{p}} \right) \\ &= x_{k+1}^{p+q} + \frac{p}{q} y_{k+1}^{p+q}. \end{aligned}$$

Using *Hölder's* inequality, we may show that

$$y_{k+1}^{p+q} \leq \frac{1}{k+1} \sum_{i=1}^{k+1} x_i^{p+q}$$

Also we have $(k+2)^p - (k+1)^p \geq p(k+1)^{p-1} \geq \frac{p}{q}(k+1)^{p-1}$.
Substituting these estimates into (9),we find

$$\sum_{i=1}^{k+1} x_i^q \Big(\sum_{j=1}^{i} x_j\Big)^p \leq \frac{q(k+1)^p}{p+q}\Big(\sum_{i=1}^{k+1} x_i^{p+q} + \frac{p}{q}\cdot\frac{1}{k+1}\sum_{i=1}^{k+1} x_i^{p+q}\Big)$$
$$\leq \frac{q(k+2)^p}{p+q}\sum_{i=1}^{k+1} x_i^{p+q}.$$

This completes the proof.

Lemma 2. *Let $u_0, u_1, u_2, \cdots, u_N$ be any sequence of real numbers, such that $u_N = 0$. If $p, q \geq 1$, then*

$$\sum_{i=n}^{N-1} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(N-n+1)^p}{p+q} \sum_{i=n}^{N} | u_i - u_{i-1} |^{p+q} . \qquad (10)$$

Proof.
Let $x_i = | u_i - u_{i-1} |$.*Then* $| u_i | \leq \sum_{j=i+1}^{N} x_j$,
so that

$$\sum_{i=n}^{N-1} | u_i - u_{i-1} |^q | u_i |^p \leq \sum_{i=n}^{N-1} x_i^q \Big(\sum_{j=i+1}^{N} x_j\Big)^p .$$

We will show by induction that

$$\sum_{i=n}^{N-1} x_i^q \Big(\sum_{j=i+1}^{N} x_j\Big)^p \leq \frac{q(N-n+1)^p}{p+q} \sum_{i=n}^{N} x_i^{p+q} . \qquad (11)$$

By Young's inequality, we have

$$x_{N-1}^q x_N^p \leq \frac{q}{p+q} x_{N-1}^{p+q} + \frac{p}{p+q} x_N^{p+q}$$
$$\leq \frac{q\cdot 2^p}{p+q} \sum_{i=N-1}^{N} x_i^{p+q} ,$$

so that (11) holds if $n = N-1$. Assume (11) holds for $n = k$, where $k < N-1$.

Since

$$\sum_{i=k-1}^{N-1} x_i^q\Big(\sum_{j=i+1}^{N} x_j\Big)^p = \sum_{i=k}^{N-1} x_i^q\Big(\sum_{j=i+1}^{N} x_j\Big)^p + x_{k-1}^q\Big(\sum_{j=k}^{N} x_j\Big)^p \qquad (12)$$

$$\leq \frac{q(N-k+1)^p}{p+q}\sum_{i=k}^{N} x_i^{p+q} + x_{k-1}^q\Big(\sum_{j=k}^{N} x_j\Big)^p.$$

Let

$$y_k = \frac{1}{N-k+1}\Big(\sum_{j=k}^{N} x_j\Big).$$

Then

$$x_{k-1}^q\Big(\sum_{j=k}^{N} x_j\Big)^p = (N-k+1)^p x_{k-1}^q y_k^p \ .$$

It follows from Young's inequality that

$$(N-k+1)^p x_{k-1}^q y_k^p \leq (N-k+1)^p\Big(\frac{q}{p+q}x_{k-1}^{p+q} + \frac{p}{p+q}y_k^{p+q}\Big).$$

Using *Hölder's* inequality, we have

$$y_k^{p+q} = \Big(\frac{1}{N-k+1}\Big)^{p+q}\Big(\sum_{j=k}^{N} x_j\Big)^{p+q} \leq \frac{1}{N-k+1}\sum_{j=k}^{N} x_j^{p+q}$$

Substituting these estimates into (12),we find

$$\sum_{i=k-1}^{N-1} x_i^q\Big(\sum_{j=i+1}^{N} x_j\Big)^p$$

$$\leq \frac{q(N-k+1)^p}{p+q}\sum_{i=k-1}^{N} x_i^{p+q} + \frac{p(N-k+1)^{p-1}}{p+q}\sum_{j=k}^{N} x_j^{p+q}$$

$$\leq \Big(\frac{q(N-k+1)^p}{p+q} + \frac{p(N-k+1)^{p-1}}{p+q}\Big)\sum_{i=k-1}^{N} x_i^{p+q}$$

$$\leq \frac{q(N-k+2)^p}{p+q}\sum_{i=k-1}^{N} x_i^{p+q}.$$

This completes the proof.

Theorem 1. *Let $u_0, u_1, u_2, \cdots, u_N \in R$ be a sequence of real numbers such that $u_0 = u_N = 0$. Then for $p, q \geq 1$, we have*

$$\sum_{i=1}^{N-1} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(N+1)^p}{2^p(p+q)} \sum_{i=1}^{N} | u_i - u_{i-1} |^{p+q} \tag{13}$$

where N is odd, and

$$\sum_{i=1}^{N-1} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(N+2)^p}{2^p(p+q)} \sum_{i=1}^{N} | u_i - u_{i-1} |^{p+q} \tag{14}$$

where N is even.

Proof. 1° N is odd:
If $N = 1$, then there is nothing to prove, so assume $N > 1$.
Let $n = \frac{N-1}{2}$. By Lemma 1, we have

$$\sum_{i=1}^{n} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(n+1)^p}{p+q} \sum_{i=1}^{n} | u_i - u_{i-1} |^{p+q} \tag{15}$$
$$= \frac{q(\frac{N-1}{2}+1)^p}{p+q} \sum_{i=1}^{\frac{N-1}{2}} | u_i - u_{i-1} |^{p+q},$$

and by Lemma 2, we have

$$\sum_{i=n+1}^{N-1} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(N-n)^p}{p+q} \sum_{i=n+1}^{N} | u_i - u_{i-1} |^{p+q} \tag{16}$$
$$= \frac{q(N - \frac{N-1}{2})^p}{p+q} \sum_{i=\frac{N+1}{2}}^{N} | u_i - u_{i-1} |^{p+q}.$$

Adding (15) and (16) to infer that

$$\sum_{i=1}^{N-1} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(N+1)^p}{2^p(p+q)} \sum_{i=1}^{N} | u_i - u_{i-1} |^{p+q}.$$

2° N is even: Let $u_{N+1} = 0$. Then it follows from (13) that

$$\sum_{i=1}^{N} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(N+2)^p}{2^p(p+q)} \sum_{i=1}^{N+1} | u_i - u_{i-1} |^{p+q} .$$

Hence

$$\sum_{i=1}^{N-1} | u_i - u_{i-1} |^q | u_i |^p \leq \frac{q(N+2)^p}{2^p(p+q)} \sum_{i=1}^{N} | u_i - u_{i-1} |^{p+q} .$$

This completes the proof.

3. Discrete Opial's inequality in n-dimensions

Lemma 3. *If z is a function from B to R such that $\nabla_1 \cdots \nabla_n z(x)$ exist and $z(0, x_2, \cdots, x_n) = z(x_1, 0, x_3, \cdots, x_n) = \cdots = z(x_1, \cdots, x_{n-1}, 0) = 0$. then*

$$\sum_B | z(x) |^p | \nabla_1 \cdots \nabla_n z(x) |^q \leq \Big(\sum_B (\prod_{i=1}^n x_i)^{p+q-1} \Big)^{\frac{p}{p+q}} \cdot \Big(\sum_B | \nabla_1 \cdots \nabla_n z(x) |^{p+q} \Big). \tag{17}$$

for $p + q > 1$, and $p, q > 0$. The notations and definitions are the same as in Theorem D.

Proof. We have

$$z(x) = \sum_{B_x} \nabla_1 \cdots \nabla_n z(y).$$

Using *Hölder's* inequality, we have

$$\begin{aligned} | z(x) | &\leq \sum_{B_x} | \nabla_1 \cdots \nabla_n z(y) | \\ &\leq \Big(\sum_{B_x} 1^{\frac{p+q}{p+q-1}} \Big)^{\frac{p+q-1}{p+q}} \\ &\quad \cdot \Big(\sum_{B_x} | \nabla_1 \cdots \nabla_n z(y) |^{p+q} \Big)^{\frac{1}{p+q}} , \end{aligned}$$

so that

$$| z(x) |^{p+q} \leq \left(\prod_{i=1}^{n} x_i\right)^{p+q-1} \left(\sum_{B_x} | \nabla_1 \cdots \nabla_n z(y) |^{p+q}\right) \tag{18}$$

Using *Hölder's* inequality again we have

$$\sum_{B} | z(x) |^{p} | \nabla_1 \cdots \nabla_n z(x) |^{q} \leq \left(\sum_{B} | z(x) |^{p+q}\right)^{\frac{p}{p+q}} \bullet \left(\sum_{B} | \nabla_1 \cdots \nabla_n z(x) |^{p+q}\right)^{\frac{q}{p+q}}$$

It follows from (18) that

$$\begin{aligned}
&\sum_{B} | z(x) |^{p} | \nabla_1 \cdots \nabla_n z(x) |^{q} \\
&\leq \left(\sum_{B} (\prod_{i=1}^{n} x_i)^{p+q-1} (\sum_{B_x} | \nabla_1 \cdots \nabla_n z(y) |^{p+q})\right)^{\frac{p}{p+q}} \\
&\quad \bullet \left(\sum_{B} | \nabla_1 \cdots \nabla_n z(x) |^{p+q}\right)^{\frac{q}{p+q}} \\
&\leq \left(\sum_{B} (\prod_{i=1}^{n} x_i)^{p+q-1} \cdot \sum_{B} | \nabla_1 \cdots \nabla_n z(x) |^{p+q}\right)^{\frac{p}{p+q}} \\
&\quad \bullet \left(\sum_{B} | \nabla_1 \cdots \nabla_n z(x) |^{p+q}\right)^{\frac{q}{p+q}} \\
&= \left(\sum_{B} (\prod_{i=1}^{n} x_i)^{p+q-1}\right)^{\frac{p}{p+q}} \\
&\quad \bullet \left(\sum_{B} | \nabla_1 \cdots \nabla_n z(x) |^{p+q}\right)
\end{aligned}$$

This completes the proof.

In case $n = 2$, and $B = [0, m_1] \times [0, m_2]$ we have

Lemma 4. *If $B_1 = [0, c_1] \times [0, c_2]$, z is a function from B_1 to R such that $\nabla_1 \nabla_2 z(x)$ exists and $z(0, x_2) = z(x_1, 0) = 0$ for $x_1 \in \{0, 1, \cdots c_1\}, x_2 \in \{0, 1, \cdots c_2\}$,then*

$$\sum_{B_1} | z(x) |^p | \nabla_1 \nabla_2 z(x) |^q \le \left(\sum_{B_1} (\prod_{i=1}^{2} x_i)^{p+q-1} \right)^{\frac{p}{p+q}} \bullet \left(\sum_{B_1} | \nabla_1 \nabla_2 z(x) |^{p+q} \right). \tag{19}$$

Proof. This follows from Lemma 3.

Lemma 5. *If $B_2 = [0, c_1] \times [c_2 + 1, m_2]$, z is a function from B_2 to R such that $\nabla_1 \nabla_2 z(x)$ exists and $z(0, x_2) = z(x_1, m_2) = 0$ for $x_1 \in \{0, 1, \cdots c_1\}, x_2 \in \{c_2 + 1, c_2 + 2, \cdots m_2\}$,then*

$$\sum_{B_2} | z(x) |^p | \nabla_1 \nabla_2 z(x) |^q \le \left(\sum_{B_2} (x_1(m_2 - x_2))^{p+q-1} \right)^{\frac{p}{p+q}} \bullet \left(\sum_{B_2} | \nabla_1 \nabla_2 z(x) |^{p+q} \right). \tag{20}$$

Proof. we have

$$-z(x) = \sum_{y_1=1}^{x_1} \sum_{y_2=x_2+1}^{m_2} \nabla_1 \nabla_2 z(y)$$

and the rest of the proof is similar to Lemma 3.

Lemma 6. *If $B_3 = [c_1 + 1, m_1] \times [0, c_2]$, z is a function from B_3 to R such that $\nabla_1 \nabla_2 z(x)$ exists and $z(m_1, x_2) = z(x_1, 0) = 0$ for $x_1 \in \{c_1 + 1, c_1 + 2, \cdots m_1\}, x_2 \in \{0, 1, \cdots c_2\}$,then*

$$\sum_{B_3} | z(x) |^p | \nabla_1 \nabla_2 z(x) |^q \le \left(\sum_{B_3} ((m_1 - x_1)x_2)^{p+q-1} \right)^{\frac{p}{p+q}} \bullet \left(\sum_{B_3} | \nabla_1 \nabla_2 z(x) |^{p+q} \right). \tag{21}$$

Proof. we have

$$-z(x) = \sum_{y_1=x_1+1}^{m_1} \sum_{y_2=1}^{x_2} \nabla_1\nabla_2 z(y)$$

and the rest of the proof is similar to Lemma 3.

Lemma 7. *If $B_4 = [c_1+1, m_1] \times [c_2+1, m_2]$, z is a function from B_4 to R such that $\nabla_1\nabla_2 z(x)$ exists and $z(m_1, x_2) = z(x_1, m_2) = 0$ for $x_1 \in \{c_1+1, c_1+2, \cdots m_1\}$, $x_2 \in \{c_2+1, c_2+2, \cdots m_2\}$, then*

$$\sum_{B_4} | z(x) |^p | \nabla_1\nabla_2 z(x) |^q \leq \left(\sum_{B_4} ((m_1 - x_1)(m_2 - x_2))^{p+q-1} \right)^{\frac{p}{p+q}} \tag{22}$$
$$\bullet \left(\sum_{B_4} | \nabla_1\nabla_2 z(x) |^{p+q} \right).$$

Proof. we have

$$z(x) = \sum_{y_1=x_1+1}^{m_1} \sum_{y_2=x_2+1}^{m_2} \nabla_1\nabla_2 z(y)$$

and the rest of the proof is similar to Lemma 3.

Combining Lemma 4,Lemma 5,Lemma 6 and Lemma 7,we have

Theorem 2. *If $B = [0, m_1] \times [0, m_2]$, m_i is odd for $i = 1, 2$. z is a function from B to R such that $\nabla_1\nabla_2 z(x)$ exists and $z(0, x_2) = z(x_1, 0) = z(m_1, x_2) = z(x_1, m_2) = 0$ for $x_1 \in \{0, 1, \cdots m_1\}$, $x_2 \in \{0, 1, \cdots m_2\}$, then*

$$\sum_{B} | z(x) |^p | \nabla_1\nabla_2 z(x) |^q \leq \prod_{i=1}^{2} \left(1 + 2^{p+q-1} + \cdots + (\frac{m_i - 1}{2})^{p+q-1} \right)^{\frac{p}{p+q}} \tag{23}$$
$$\bullet \left(\sum_{B} | \nabla_1\nabla_2 z(x) |^{p+q} \right).$$

Proof.

Let $c_i = \frac{m_i-1}{2}$ $i = 1, 2$. Then by (19), (20), (21) and (22) we have

$$\sum_{B_m} | z(x) |^p | \nabla_1 \nabla_2 z(x) |^q \leq \prod_{i=1}^{2} \left(1 + 2^{p+q-1} + \cdots + (\frac{m_i - 1}{2})^{p+q-1}\right)^{\frac{p}{p+q}} \tag{24}$$

$$\bullet \left(\sum_{B_m} | \nabla_1 \nabla_2 z(x) |^{p+q}\right), (m = 1, 2, 3, 4).$$

Therefore

$$\begin{aligned} \sum_{B} | z(x) |^p | \nabla_1 \nabla_2 z(x) |^q &= \sum_{m=1}^{4} \sum_{B_m} | z(x) |^p | \nabla_1 \nabla_2 z(x) |^q \\ &\leq \sum_{m=1}^{4} \left(\prod_{i=1}^{2} (1 + 2^{p+q-1} + \cdots + (\frac{m_i - 1}{2})^{p+q-1})^{\frac{p}{p+q}} \right. \\ &\quad \left. \bullet \sum_{B_m} | \nabla_1 \nabla_2 z(x) |^{p+q}\right). \\ &= \prod_{i=1}^{2} \left(1 + 2^{p+q-1} + \cdots + (\frac{m_i - 1}{2})^{p+q-1}\right)^{\frac{p}{p+q}} \\ &\quad \bullet \left(\sum_{B} | \nabla_1 \nabla_2 z(x) |^{p+q}\right). \end{aligned}$$

This completes the proof.

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