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ON GRAPH REPRESENTATION FOR CO-DIAGONAL GROUPS

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Abstract. For a co-diagonal Butler group G with co-representing graph, we examine the conditions under which pure subgroups and torsion-free quotients of G have quasi-co-representing graphs.

1. Introduction

For a rank 1 torsion-free abelian group X, let type(X) denote the isomorphism class of X called a type. For any element g of a finite rank torsion-free group G, we let $\text{type}_G(g)$ denote the type($< g >_*$), where $< g >_*$ is the rank-1 pure subgroup generated by g. If τ is a type, let $G(\tau) = \{x \in G | \text{type}_G(x) \geq \tau\}$ and $G^*(\tau) = \Sigma\{G(\sigma)|\sigma > \tau\}$. $G[\tau]$ is defined to be $\cap \{\ker f | f \in \text{Hom}(G, X_\tau)\}$, where X_τ is a subgroup of Q with type$(X_\tau) = \tau$. Also, define $G^*[\tau] = \cap\{G[\sigma]|\sigma < \tau\}$. An exact sequence of torsion-free abelian groups $0 \to A \to B \to C \to 0$ is co-balanced if $0 \to A/A[\tau] \to B/B[\tau]$ is pure exact for each type τ.

Butler groups are pure subgroups of finite rank completely decomposable groups. This class of Butler groups plays an important role in the theory of finite-rank torsion-free abelian groups. In this paper we shall be concerned with co-diagonal groups.

Co-diagonal Butler groups are groups which are quasi-isomorphic to $G < A >= (A_1 \oplus \ldots \oplus A_n)/ < (1, \ldots, 1) >_*$ for some n-tuple $A = (A_1, \ldots, A_n)$ of subgroups of Q. The group $G < A >$ is the dual of $G(A) = \{(a_1, \ldots, a_n) \in$

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$A_1 \oplus \ldots \oplus A_n | \Sigma_{i=1}^{n} a_i = 0\}$ [5]. This class of co-diagonal groups is presented in [1] and some of their properties can interpreted through graph representation [2].

Let $(A_1, \ldots, A_n)$ be an n-tuple of subgroups of Q and T be a connected labelled graph with vertices $A_1, \ldots, A_n$ and each edge ij in T labelled by a type $\tau^{ij} = \text{type}(A^{ij} = A_i + A_j) = \sup\{\text{type}(A_i), \text{type}(A_j)\}$.

The graph T is a co-representing graph for $G = G < A_1, \ldots, A_n >= (A_1 \oplus \ldots \oplus A_n)/X$ with $X =< (1, \ldots, 1) >_*$ if

(1) $0 \to G \xrightarrow{\delta_T} D_T$ is co-balanced exact.

(2) For each type τ, $\delta_T^{\tau}(G/G[\tau]) = G^{\tau} \oplus G^* < \tau >$,

where $D_T = \oplus\{A^{ij} | ij \in T\}$, $\delta_T(x_i + X) = \oplus\{x_{ij} | x_{ij} = x_i$ if $i < j$ and $x_{ij} = -x_i$ if $j < i$, $ij \in T\}$ with $x_i \in A_i$, $G^{\tau} = \oplus\{A^{ij} | \tau^{ij} = \tau, ij \in T\}$, $G^* < \tau >= \Sigma_k \delta_{T^*<\tau>}((A_k + X)/X)$ with $\delta_{T^*<\tau>}((A_k + X)/X) = \delta_T((A_k + X)/X) \cap D_{T^*<\tau>}$ and $T^* < \tau >= \{ij \in T | \tau^{ij} < \tau\}$.

The graph T is a quasi-co-representing graph for a Butler group G if T is a co-representing graph for some $G < A_1, \ldots, A_n >$ quasi-isomorphic to G.

In [2], we have developed a realization theorem for graph which are finite, connected, and with edges labelled by types. Furthermore, for each co-diagonal Butler group $G < A >$, we can construct its co-representing graph.

In the paper, we are going to examine the conditions under which pure subgroups and torsion-free quotients of groups with co-representing graphs have quasi-co-representing graph. In particular, if G is a co-diagonal Butler group with co-representing graph, then $G(\tau)$, $G[\tau]$, $< G^*(\tau) >_*$, $G/G[\tau]$ and $G^*[\tau]/G[\tau]$ have quasi-co-representing graphs for each type τ.

2. Main Results

Suppose that T is a co-representing graph for $G = G < A_1, \ldots, A_n >$, and that S is a subset of the set of edges of T. The equivalence relation on the vertices of T determined by S is defined by $i \approx i$, and for $i \neq j$, $i \approx j$ whenever i and j are connected by a path in $T \backslash S$. Define $M < S >= \{g \in G | \delta_T(g) \in D_S\}$. Note that $M < S >$ is a pure subgroup of G which is isomorphic to $\delta_T(G) \cap D_S$.

Theorem 2.1. *Assume that T is a co-representing graph for $G = G < A_1, \ldots, A_n >$, $S \subset T$, and $V_1, \ldots, V_k$ is the set of equivalence classes of vertices of T determined by S. Then*

(a) $M < S > \cong G < B_1, \ldots, B_k >$, where $B_i = \cap\{A_j | j \in V_i\}$ if $k > 1$, and $M < S >$ is 0 if $k = 1$.

(b) $G/M < S > \cong G_1 \oplus \ldots \oplus G_k$, where $G_i = G < A_j | j \in V_i >$ if $|V_i| > 1$, and $G_i = 0$ if $|V_i| = 1$.

Proof. (a) First, observe that $M < S >= \{g = (a_1, \ldots, a_n) + X \in G | a_t = a_{t'} \in B_i, t, t' \in V_i, i = 1, \ldots, k\}$.

To see this, let $g = (a_1, \ldots, a_n) + X \in G$. Then $\delta_T(g) = \oplus\{a_i + (-a_j) | i < j, ij \in T\}$. If $a_t = a_{t'} \in B_i$ for $t, t' \in V_i$, $i = 1, \ldots, k$, then the ij-coordunate $(\delta_T(g))_{ij}$ of $\delta_T(g)$ is 0 for each $ij \in T \setminus S$ since i and j belong to the same class. Hence, $\delta_T(g) \in D_S$.

Conversely, if $(\delta_T(g))_{ij} = 0$ for each $ij \in T \setminus S$, then $a_i = a_j$ for each $ij \in T \setminus S$. Hence, $a_t = a_{t'}$ for $t, t' \in V_i$, $i = 1, \ldots, k$.

Now define $f : G < B_1, \ldots, B_k > \rightarrow M < S >$ by $f((b_1, \ldots, b_k) + X) = (a_1, \ldots, a_n) + X$, where $a_t = b_i$ for $t \in V_i$, $i = 1, \ldots, k$.

It is easy to see that f is epic.

If $f((b_1, \ldots, b_k) + X) = (a_1, \ldots, a_n) + X = 0$, then $a_1 = \ldots = a_n$. Hence, $b_1 = \ldots = b_k$. and $(b_1, \ldots, b_k) + X = 0$.

This proves that f is monic.

Therefore, f is an isomorphism and $G < B_1, \ldots, B_k > \cong M < S >$.

(b) we next show that $G/M < S > \cong G_1 \oplus \ldots \oplus G_k$ by defining a function $h : G \rightarrow G_1 \oplus \ldots \oplus G_k$, where $h((a_1, \ldots, a_n) + X) = \oplus_{i=1}^{k}\{(a_{i_1}, \ldots, a_{i_m}) + X | i_s \in V_i\}$. Then it is easy to see that h is epic with ker $h = M < S >$. Hence, (b) is proved.

Examples of Theorem 2.1 are included in the following corollary.

Corollary 2.2. *Suppose that T is a co-representing graph for $G = G < A_1, \ldots, A_n >$ and τ is a type. Then*

(a) $G(\tau) = M < S(\tau) >$, *where* $S(\tau) = \{ij \in T | \tau^{ij} \geq \tau\}$.

(b) $G[\tau] = M < S[\tau] >$, *where* $S[\tau] = [ij \in T | \tau^{ij} \not\leq \tau\}$.

(c) $G(\tau)$, $G[\tau]$, $< G^*(\tau) >_*$, $G/G[\tau]$ *and* $G^*[\tau]/G[\tau]$ *have quasi-co-representing graphs induced by* T.

Proof. Suppose that $g \in G$ and $\text{type}_G(g) \geq \tau$. Then $\text{type}_G(g) = \text{type}_{D_T}(\delta_T(g)) = \inf\{\text{type}(A^{ij}) | \Pi_{ij}\delta_T(g) \neq 0\} \geq \tau$. where $\Pi_{ij} : D_T \to A^{ij}$ is the projection of D_T onto A^{ij}, noting that $0 \to G \xrightarrow{\delta_T} D_T$ is co-balanced exact, hence pure exact. Hence, $\delta_T(g) \in \oplus\{A^{ij} | ij \in S(\tau)\}$.

Conversely, let $\delta_T(g) \in \oplus\{A^{ij} | ij \in S(\tau)\}$ for some $g \in G$. Then $\text{type}_{D_T}(\delta_T(g)) \geq \tau$, since $\text{type}(A^{ij}) \geq \tau$ for each $ij \in S(\tau)$. Hence, $\text{type}_G(g) = \text{type}_{D_T}(\delta_T(g)) \geq \tau$ and $g \in G(\tau)$.

(b) First, note that $g[\tau] =< G(\sigma) | \sigma \not\leq \tau >_*$, by Proposition 1.9 in [3]. Suppose that $g \in G$ and $\text{type}_G(g) \not\leq \tau$. Then $\text{type}_G(g) = \text{type}_{D_T}(\delta_T(g)) = \inf\{\text{type}(A^{ij}) | \Pi_{ij}\ \delta_T(g) \neq 0\} \not\leq \tau$. Hence, $\delta_T(g) \in \oplus\{A^{ij} | ij \in S[\tau]\}$. Otherwise $\text{type}_{D_T}(\delta_T(g)) \leq \tau$, because if $\text{type}(A^{ij}) \leq \tau$ for some $ij \in T$ with $\Pi_{ij}\delta_t(g) \neq 0$, then $\text{type}_{D_T}(\delta_T(g)) \leq \text{type}(A^{ij}) \leq \tau$.

For eaach $x \in G[\tau]$, there exists $0 \neq m \in Z$ such that $mx = g_1 + \ldots + g_k$ with $\text{type}_G(g_i) \not\leq \tau$. Then $\delta_T(mx) \in \oplus\{A^{ij} | ij \in S[\tau]\}$, since each $\delta_T(g_i) \in \oplus\{A^{ij} | ij \in S[\tau]\}$. It follows that $\delta_T(x) \in \oplus\{A^{ij} | ij \in S[\tau]\}$, because $\delta_T(G)$ is pure in D_T. This proves that $\delta_T(G[\tau]) \subset \oplus\{A^{ij} | ij \in S[\tau]\}$.

Conversely, if $\delta_T(g) \in \oplus\{A^{ij} | ij \in S[\tau]\}$, then $g \in G[\tau]$, since $\delta_T^{\tau}(G/G[\tau]) = G^{\tau} \oplus G^* < \tau > \subset \oplus\{A^{ij} | ij \in T, \tau^{ij} \leq \tau\}$. The reason is that if $g \notin G[\tau]$, then $g + G[\tau] \neq 0$. It follows that $\delta_T^{\tau}(g + G[\tau]) = \delta_T(g) + D_T[\tau] \neq 0$ in $D_T/D_T[\tau] = \oplus\{A^{ij} | ij \in T\,\tau^{ij} \leq \tau\}$, noting that $0 \to G/G[\tau] \xrightarrow{\delta_T^{\tau}} D_T/D_T[\tau]$ is pure exact. This means that there exists $ij \in T$ with $\tau^{ij} \leq \tau$ such that $(\delta_T(g))_{ij} \neq 0$. Then $\delta_T(g) \notin \oplus\{A^{ij} | ij \in S[\tau]\}$, a contradiction.

(c) By (a), (b) and Theorem 2.1, $G(\tau)$ and $G[\tau]$ have quasi-co-representing graphs. It follows that $< G^*(\tau) >_* = G(\tau)[\tau]$ has a quasi-co-representing graph.

To show that $G/G[\tau]$ has quasi-co-representing graph, it is sufficient to prove that if $V_1, \ldots, V_k$ are the equivalence classes of vertices determined by $S[\tau]$, then

$|V_i| > 1$ for at most one i. In this case, $G/G[\tau] \cong G < A_j | j \in V_i >$ has a quasi-co-representing graph by Theorem 2.1.(b). To see that this is indeed the case, note that $V_1 = \{i | ij \in T \setminus S[\tau] \text{ for some } j\}$ is an equivalence class, since $T < \tau >= \{ij \in T | \tau^{ij} \leq \tau\}$ is connected by Corollary 2.9.(a) in [2]. On the other hand, if $i \notin V_1$, then the equivalence class of i is $\{i\}$, since i does not lie on an edge of $T \setminus S[\tau]$.

Finally, $G/G[\tau] = G^*[\tau]/G[\tau] \oplus G'$ where $G' \cong G/G^*[\tau]$ and $G^*[\tau]/G[\tau] = (G/G[\tau])\ (\tau)$ by Theorem 1.7. (c) in [3]. Hence, $G^*[\tau]/G[\tau]$ has a quasi-co-representing graph.

Actually, $G^*[\tau]/G[\tau]$ is τ-homogeneous completely decomposable by Theorem 1.7.(c) in [3], which belongs to Ω and has a quasi-co-representing graph by Theorem 2.7 in [2], where Ω is the class of all torsion free abelian groups quasi-isomorphic to Butler groups which are homomorphic images of completely decomposable groups with rank-1 kernels.

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