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SKEW POLYNOMIAL RINGS SATISFYING R-BND PROPERTY

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Abstract. In this paper we show that, a prime right Noetherian ring A satisfies $T(A) = n < \infty$ iff $A[x, \sigma]$ satisfies r-Bnd $(n+1)$.

O. Introduction

In [6] Robson has found a relation between the Krull dimension (in the sense of Rentschler, [5]) and the upper bound of the number of generators of right ideals in polynomial rings over simple right Noetherian rings. Also Stafford in [8] studied the relation between a ring A and its polynomial ring $A[x]$ if one of them satisfies r-Bnd property. Here we extend Robson's result to the Ore extension over simple right Noetherian rings and study the relation between the two properties $T(A) = n < \infty$ and the r-Bnd(n) of the skew polynomial ring $A[x, \sigma]$.

I. Definitions and Basic Concepts

All rings here are with identity and all modules are unitary. A ring A is said to be $T(A) = n < \infty$, if every finitely generated torsion right A-module can be generated by n elements. The ring A (the module M_A) is said to be r-band(n), if every right ideal of A (submodule of M_A) can be generated by n elements. A right A-module M is said to be completely faithful, if each nonzero subfactor module is faithful, it is clear that each module over simple rings is completely

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faithful. Consider the Ore extension ring of A, that is the ring $R = A[x,\sigma,\delta]$, where σ is an automorphism of A and δ is a σ-derivation of A, where addition is component wise and multiplication is given as

$$(ab)x = x\sigma(ab) + \delta(ab) \text{ and } \delta(ab) = \sigma(a) \quad \delta(b) + a\delta(b).$$

The right ideal I of A is called σ-ideal if $\sigma(I) \subseteq I$, it is well known that if A is right Noetherian ring, then $\sigma(I) = I$. The ideal I is called σ-prime ideal of A, if whenever J, K are σ-ideals of A such that $JK \subseteq I$, then $J \subseteq I$ or $K \subseteq I$. A is called σ-prime ring if (0) is a σ-prime ideal of A. The ideal I is called σ-maximal of A, if there is no proper σ-ideal J such that $I \subset J \subset A$.

II. Preliminary Results and Remarks

1. Let $R = A[x,\sigma,\delta]$, where A is any ring, σ is an automorphism of A and δ is a σ-derivation of A. Let S be a multiplicative set of regular elements of A, such that $\sigma(S) \subseteq S$ and S satisfies the right Ore condition. Let $Q = AS^{-1}$, then

i) σ and δ can be extended in a unique manner to an automorphism σ' of Q and to a σ'-derivation δ' of Q.

ii) S is the multiplicative set of regular elements of R, R satisfies the right Ore condition for S and $RS^{-1} = Q[x,\sigma',\delta']$ ([2], Theorem 7.1.2).

2. If A is a simple right Artinian ring, then $R = A[x,\sigma,\delta]$ is a principal right ideal ring, where σ is an automorphism of A and δ is a σ-derivation ([2], Corollary 6.2.2).

3. Suppose $n < \infty$ and A is a ring with $K(A) > n$. If M is a completely faithful Noetherian right A-module, such that $K(M) = n$, then M can be generated by $n+1$ elements [7].

4. If A is a simple right Noetherian ring with $K(A) = n$, then any right ideal of A can be generated by $n+1$ elements [7].

5. (i) Let A be a ring with right Krull dimension and $c \in S$ (the m-set of regular elements), then $K\{(A/cA)_{\mathbf{A}}\} < K(A_{\mathbf{A}})$ ([4], Lemma 6.3.9 p.189).

(ii) If $M_{\mathbf{A}}$ is finitely generated, then $K(M) \leq K(A_{\mathbf{A}})$ ([4], Lemma 6.2.5 p.131).

(iii) If $N_{\mathbf{A}}$ is a submodule of $M_{\mathbf{A}}$, then $K(M) = \sup\{K(N), K(M/N)\}$ ([4], Lemma 6.2.4 p.180).

(iv) Let $M_{\mathbf{A}}$ have Krull dimension and also be the sum of submodules each of which has Krull dimension $\leq \alpha$, then $K(M) \leq \alpha$ ([4], Lemma 6.2.14 p.184).

6. Let I be a nonzero σ-ideal of the σ-prime right Noetherian ring A, then $I \cap S \neq \phi$, where S is the m-set of regular elements ([1], proposition I.12 p.I.14).

7. The following are equivalent for a ring A, with an automorphism σ of A and a σ-derivation δ of A.

(i) A is right Noetherian.

(ii) $A[x, \sigma, \delta]$ is right Noetherian ([3], Theorem 2.2.15).

Lemma 1. *Let A be a prime right Goldie ring and $R = A[x, \sigma, \delta]$, where σ is an automorphism of A and δ is a σ-derivation. If I is a right ideal of $R = A[x, \sigma, \delta]$, then I contains an element g such that $I/J = I/gA[x, \sigma, \delta]$ is a torsion right A-module.*

Proof. Since, A is a prime right Goldie ring, then A has a right quotient ring Q which is simple Artinian. By {remark 1(ii)} the automorphism σ of A and the σ-derivation δ of A can be extended in a unique manner to σ' of Q and a σ'-derivation δ' of Q. Consider the Ore extension ring $Q[x, \sigma', \delta']$, then by {remark 1(ii)} $Q[x, \sigma', \delta'] = AS^{-1}[x, \sigma', \delta'] = A[x, \sigma, \delta]S^{-1}$. Using (remark 2) $Q[x, \sigma', \delta']$ is a principal right ideal ring. Let I be a right ideal of $R = A[x, \sigma, \delta]$, then $I_{\mathbf{S}} = IQ[x, \sigma', \delta'] = \{ks^{-1} | k \in I,\ s \in S\}$. Since $I_{\mathbf{S}}$ is an ideal in $Q[x, \sigma', \delta']$, then $I_{\mathbf{S}} = hA[x, \sigma, \delta]_{\mathbf{S}}$, where $h \in I_{\mathbf{S}}$. Since, S satisfies the right Ore condition, then $I_{\mathbf{S}} = gs^{-1}A[x, \sigma, \delta]_{\mathbf{S}} = gA[x, \sigma, \delta]_{\mathbf{S}}$, where $g \in I\Delta_r A[x, \sigma, \delta]$. Let $J = gA[x, \sigma, \delta]$ and consider the right A-module $M = I/J$, this is a torsion right A-module. Since, $I \subseteq I_{\mathbf{S}} = gA[x, \sigma, \delta]_{\mathbf{S}} = \{ks^{-1} | k \in I, s \in S\}$, then each $i \in I$ can be written as $i = gf$, where $f \in A[x, \sigma, \delta]_{\mathbf{S}}$. Thus $i = gms_1^{-1}$ where $m \in A[x, \sigma, \delta]$ and $s_1 \in S$. Accordingly, $is_1 = gm \in J = gA[x, \sigma, \delta]$ and $M = I/J$ is a torsion right A-module.

Lemma 2. *Let A be a prime right Noetherian ring which satisfies $T(A) = n < \infty$, then the Ore extension ring $R = A[x, \sigma, \delta]$ of A satisfies r-Bnd $(n+1)$.*

Proof. Let I be a nonzero right ideal of R. Since, A is right Noetherian, then R is right Noetherian by (remark 7) and $I = \Sigma_{i=1}^{k} g_i R$ say $k > n+1$. Using (Lemma 1), there exists a nonzero element $g \in I$ such that $I/J = I/gR$ is a torsion right A-module. Therefore, for each $g_i \in I$, there exists $r_i \in S$ (the m-set of regular elements) such that $g_i r_i \in J$. Noe, if we define R-homomorphism,

$$\phi : \sum_{i=1}^{k} \oplus (g_i R/g_i r_i R) \rightarrow \sum_{i=1}^{k} g_i R / \sum_{i=1}^{k} g_i r_i R \qquad \text{as}$$

$$\phi : (g_1 a_1 + H_1, \ldots, g_k a_k + H_k) \longmapsto (g_1 a_1 + \ldots + g_k a_k) + H, \text{ where}$$

$H_i = g_i r_i R$ and $H = \Sigma_{i=1}^{k} g_i r_i R$, then it is easily verified that ϕ is a well defined onto R-homomorphism. Also since, $g_i r_i \in J$ for each $i = 1, \ldots, k$, then $g_i r_i R \subseteq J$ and $\Sigma_{i=1}^{k} g_i r_i R \subseteq J$, thus $\Phi : \Sigma_{i=1}^{k} g_i R / \Sigma_{i=1}^{k} g_i r_i R \rightarrow \Sigma_{i=1}^{k} g_i R / gR = I/J$, is onto.

Consequently, $\tau : \Sigma_{i=1}^{k}{}^{\oplus} g_i R / g_i r_i R \rightarrow I/J$, where $\tau = \Phi \circ \phi$ is also onto. Moreover, if we define R-homomorphism-:
$\Theta : \Sigma_{i=1}^{k} \oplus (R/r_i R) \rightarrow \Sigma (g_i R/g_i r_i R)$ as $\Theta : (a_1 + H_1', \ldots, a_k H_k') \longmapsto (g_1 a_1 + H_1, \ldots, g_k a_k + H_k)$, where $H_i' = r_i R$, then it is easily shown that Θ is well defined onto R-homomorphism.

Summerizing I/J is the homomorphic image of $\Sigma_{i=1}^{k}{}^{\oplus} R/r_i R$. Since, A satisfies $T(A) = n < \infty$, and $\bigoplus_{i=1}^{k} A/r_i A$ is finitely generated torsion right A-module, hence $\bigoplus_{i=1}^{k} A/r_i A$ can be generated by n elements as A-module. Therefore, $\bigoplus_{i=1}^{k} R/r_i R$ can also be generated by n elements as R-module. Since I/J is its homomorphic image, then it is generated as an R-module by n elements. Hence, I is generated by $n+1$ elements and the Lemma is proved.

The following result shows how can the right (left) Krull dimension [6] play an important role in determining the upper bound of the number of generators of the right (left) ideals in Ore extension rings.

Proposition 3. *Let A be a simple right Noetherian ring and $K(A_A) = n$, then both A and $R = A[x, \sigma, \delta]$ satisfies r-Bnd $(n + 1)$.*

Proof. Since, A is a simple right Noetherian ring and $K(A_A) = n$, then by (remark 4) A satisfies r-Bnd $(n + 1)$. Also, using (remark 7) R is right Noetherian. Then by the same argument used in (Lemma 1) one can easily check that any nonzero right ideal I of R contains a nonzero element g and the right R-submodule $J = gR$ such that I/J is a torsion right A-module. Let $I = \Sigma_{i=1}^{k} a_i R$, where $k > n + 1$, since I/J is a torsion right A-module, then for each a_i there exists $r_i \in S$ such that $a_i r_i \in J$. Also, as in (Lemma 2) it can be easily verified that I/J is the homomorphic image of $\Sigma_{i=1}^{k}{}^{\oplus}(A/r_i A)[x, \sigma, \delta]$. Since, each r_i is regular and $K(A) = n$, then by {remark 5(i)} $K(A/r_i A) < n$ for each $i = 1, \ldots, k$. Consider the right A-module $M = \Sigma_{i=1}^{k} A/r_i A$, since M is a finitely generated A-module, then $K(M) \leq n$ by {remark 5(ii)} and since, M is the sum of submodules each of Krull dimension $< n$, then by {remark 5(iv)} $K(M) < n$. Since, M is the homomorphic image of $\bigoplus_{i=1}^{k} A/r_i A$ we get that $K(\bigoplus_{i=1}^{k} A/r_i A) \leq K(M) < n$ by {remark 5(iii)}. Since A is simple and $K(\bigoplus_{i=1}^{k} A/r_i A) < n$, then by (remark 3) $\bigoplus_{i=1}^{k} A/r_i A$ can be generated by n elements as A-module. Conequently, $\bigoplus_{i=1}^{k} A/r_i A[x, \sigma, \delta]$ can be generated by n elements as R-module. Hence, I/J can be generated by n elements as a homomorphic image of $\bigoplus_{i=1}^{k} A/r_i A[x, \sigma, \delta]$. Then I can be generated by $n + 1$ elements.

Propossition 4. *Let A be a σ-prime right Noetherian ring satisfies $T(A) = n < \infty$. Then A is σ-simple.*

Proof. Suppose that A is not σ-simple, then it contains a proper σ-ideal p, take M to be the direct sum of m copies of A/p where $m > n$. Since A is a σ-prime right Noetherian ring and p is a nonzero σ-ideal, then by (remark 6) $p \cap S \neq \phi$. The regular elements that belong to p annihilate all components of

$M = (A/p)^m$. Thus, $M = (A/p)^m$ is a finitely generated torsion right A-module which can't be generated by less than $m > n$ elements which contradicts our asssumption. Thus A is a σ-simple ring.

Lemma 5. *Let A be a σ-prime right Noetherian ring such that $A[x,\sigma]$ satisfies r-Bnd(n), then A satisfies $T(A) = n < \infty$ and A is σ-simple.*

Proof. Consider a finitely generated torsion right A-module $M = \Sigma_{i=1}^m a_i A$, $m > n$. So, for each a_i there exists $r_i \in S$ (the m-set of regular elements) such that $a_i r_i = 0$. Let α_i be an A-homomorphism: $A \to a_i A$, since $a_i r_i = 0$ then $r_i A \subseteq ker\alpha_i$ for each $i = 1, \ldots, m$ and we have an onto A-homomorphism: $A/r_i A \to A/ker\,\alpha_i \simeq a_i A$. Now, consider the A-homomorphism $\phi : \oplus_{i=1}^m A/ker\,\alpha_i \to \Sigma_{i=1}^m a_i A = M$ defined by $\phi : (b_1 + ker\,\alpha_1, \ldots, b_m + ker\,\alpha_m) \longmapsto \Sigma_{i=1}^m a_i b_i.\phi$ is well defined, since if $(b_1 + ker\,\alpha_1, \ldots, b_m + ker\,\alpha_m) = 0$, then each $b_i \in ker\,\alpha_i$(i.e. $a_i b_i = 0$) hence, $\Sigma_{i=1}^m a_i b_i = 0$ and it is clear that ϕ is onto. Let $\Phi : \oplus_{i=1}^m A/r_i A \to \oplus_{i=1}^m A/ker\,\alpha_i$ be an A-homomorphism defined by $\Phi : (b_1 + r_1 A, \ldots, b_m + r_m A) \longmapsto (b_1 + ker\,\alpha_1, \ldots, b_m + ker\,\alpha_m)$ it clear that Φ is a well defined and onto A-homomorphism. So, M is a homomorphic image of $\oplus_{i=1}^m A/r_i A$. Define an A-homomorphism $\mu : \oplus_{i=1}^m A/r_i A \to \oplus_{i=1}^m A/\sigma^{m-i+1}(r_i)A$ by $\mu : (b_1 + r_1 A, \ldots, b_m + r_m A) \longmapsto (\sigma^m(b_1 + r_1 A), \ldots, \sigma(b_m + r_m A)) = (\sigma^m(b_1) + \sigma^m(r_1 A), \ldots, \sigma(b_m) + \sigma(r_m)A)$ μ is well defined since $\mu(b_1 + r_1 A, \ldots, b_m + r_m A) = 0$ iff $b_i \in r_i A$, $i = 1, \ldots, m$ which is equivalent to $\sigma^{m-i+1}(b_i) = \sigma^{m-i+1}(r_i)a' \in \sigma^{m-i+1} r_i A$, where $a' \in A$. Thus, $0 = (b_1 + r_1 A, \ldots, b_m + r_m A) \longmapsto (\overline{0}, \ldots, \overline{0})$ also, μ is onto. Hence, $\oplus_{i=1}^m A/r_i A \simeq \oplus_{i=1}^m A/\sigma^{m-i+1}(r_i)A = N$. Give N an $A[x,\sigma]$-module structure by defining $Nx = 0$. Let I be a nonzero right ideal of $A[x,\sigma]$ given by $I = x^m A[x,\sigma] + r_1 x^{m-1} A[x,\sigma] + \ldots + (r_1 \ldots r_{m-1}) x A[x,\sigma]$.

Define an $A[x,\sigma]$-homomorphism $\Omega : I \to N$ as follows $\Omega : (x^m f_1(x) + r_1 x^{m-1} f_2(x) + \ldots + r_1 \ldots r_{m-1} x f_m(x)) \longmapsto (f_1(0) + H_1, f_2(0) + H_2, \ldots, f_m(0) + H_m)$ where $H_i = \sigma^{m-i+1}(r_i)A$. Ω is a well defined since if $g(x) \in I$, then g can be written as $g(x) = x^m \alpha_0^{(1)} + r_1 x^{m-1}(\alpha_0^{(2)} + x\alpha_1^{(2)}) + \ldots + r_1 \ldots r_{m-1} x(\alpha_0^m + \ldots + x^{m-1}\alpha_{m-1}^{(m)}) +$ terms of higher degrees, where $\sigma_j^{(i)}$ is the coefficient of x^j in the polynomial $f_i(x)$. Thus $g(x) = x^m\{\alpha_0^{(1)} + \sigma^m(r_1)\alpha_1^{(2)} + \ldots + \sigma^m(r_1) \ldots \sigma^m(r_{m-1})$

$\alpha_{m-1}^{(m)}\} + r_1 x^{m-1}\{\alpha_0^{(2)} + \sigma^{m-1}(r_2)\alpha_1^{(3)} + \ldots + \alpha^{m-1}(r_2 \ldots r_{m-1})\alpha_{m-2}^{(m-1)}\} + \ldots +$ $(r_1 \ldots r_m)x\alpha_0^m$. But $g(x) = 0$ means that $\alpha_0^{(1)} \in \sigma^m r_1(A)$, $\alpha_0^{(2)} \in \sigma^{(m-1)}(r_2)A$, $\ldots, \alpha_0^{(m)} = 0$ which gives that $f_1(0) = \alpha_0^{(1)} \in H_1$, $f_2(0) = \alpha_0^{(2)} \in H_2, \ldots, f_m(0)$ $= \alpha_0^{(m)} = 0$ i.e. $\Omega g(x) = 0$. That Ω is an $A[x,\sigma]$-module homomorphism follows directly if we notice that $\Omega(g(x)\alpha) = n\alpha$ and $\Omega(g(x)x) = nx = 0$ where $\alpha \in A$ and $n = \Omega(g(x))$. Also, it is evident that Ω is onto. Therefore, N is a homomorphic image of I as $A[x,\sigma]$-module. Since, $A[x,\sigma]$ satisfies r-Bnd(n), then I can be generated by n elements. Consequently, N is generated as $A[x,\sigma]$-module by n elements. But since, $Nx = 0$, the same n elements will generate N as an A-module. Thus M, as a homomorphic image of N, is generated by n elements. Consequently, A satisfies $T(A) = n < \infty$ and by (Lemma 4) A is σ-simple.

Now if we put $\delta = 0$ in (Lemma 2) then it follows, using the above proposition that

Theorem 6. *If A is a prime right Noetherian ring, then the following conditions are equivalent:*

1) *A satisfies $T(A) = n < \infty$*

2) *$A[x,\sigma]$ satisfies r-Bnd($n = 1$).*

Proposition 7. *Let A be a ring such that $A[x,\sigma]$ satisfies r-Bnd(n), then $T(A/p) = n$ for each σ-prime ideal p of A.*

Proof. Since $A[x,\sigma]$ satisfies r-Bnd(n), then $A[x,\sigma]$ is right Noetherian. So, by (remark 7) A is right Noetherian. Since p is a σ-prime ideal, then A/p is a σ'-prime ring, where σ' is an automorphism of A/p induced by σ and $A/p[x,\sigma'] \simeq A[x,\sigma]/p[x,\sigma]$. Since, $A[x,\sigma]$ satisfies r-Bnd(n), then $A[x,\sigma]/p[x,\sigma]$ satisfies r-Bnd(n). Hence, $A/p[x,\sigma']$ satisfies r-Bnd(n). Using (Lemma 5) A/p satisfies $T(A/p) = n < \infty$.

Corollary 8. *Let A be a ring such that $A[x,\sigma]$ satisfies r-Bnd(n), then all σ-prime ideal of A are σ-maximal.*

Proof. This follows directly from propositions 4 and 7.

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