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A GENERALIZATION OF INVERSE SCHWARZ'S INEQUALITY

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1. Introduction

In 1915, Frank and Pick [4] proved the following inverse Schwarz's inequality: If u and v are nonnegative and concave functions on the interval $[0,1]$, then

$$\int_0^1 u(t)v(t)dt \geq \frac{1}{2}\Big(\int_0^1 u^2(t)dt\Big)^{\frac{1}{2}}\Big(\int_0^1 v^2(t)dt\Big)^{\frac{1}{2}}. \tag{1.1}$$

Bellman [2] generalized (1.1) into the following form:

$$\int_0^1 u(t)v(t)dt \geq \frac{(A+1)^{\frac{1}{A}}(B+1)^{\frac{1}{B}}}{6}\Big(\int_0^1 u^A(t)dt\Big)^{\frac{1}{A}}\Big(\int_0^1 v^B(t)dt\Big)^{\frac{1}{B}}, \tag{1.2}$$

where $1 < A, B < \infty$ with $\frac{1}{A}+\frac{1}{B}=1$.
We note that the inequality (1.1) is the special case of (1.2) when $A = B = 2$.

Recently, Alzer [1] established the following generalization of (1.1):

$$\int_0^1 u^p(t)v^q(t)dt \geq \Big(\frac{2\sqrt{(2p+1)(2q+1)}}{(p+1)(q+1)}-1\Big)\Big(\int_0^1 u^{2p}(t)dt\Big)^{\frac{1}{2}}\Big(\int_0^1 v^{2q}(t)dt\Big)^{\frac{1}{2}}, \tag{1.3}$$

where $p > 0, q > 0$.
We note that the inequality (1.1) is the special case of (1.3) when $p = q = 1$.

Our aim of this article is to generalize the inequality (1.3) in several forms.

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2. Preliminaries

Lemma 1. (Alzer [1]) *Let u, v and w be integrable functions on $[0,1]$ and $\int_0^1 w^2(t)dt \neq 0$. Then*

$$\frac{1}{2}[(\int_0^1 u^2(t)dt)^{\frac{1}{2}}(\int_0^1 v^2(t)dt)^{\frac{1}{2}} + \int_0^1 u(t)v(t)dt] \geq \frac{\int_0^1 u(t)w(t)dt \cdot \int_0^1 v(t)w(t)dt}{\int_0^1 w^2(t)dt}.$$

Lemma 2. (Berwald [3]) *Let u be nonnegative and concave on $[0,1]$. If $0 < p \leq q$, then*

$$[(q+1)\int_0^1 u^q(t)dt]^{\frac{1}{q}} \leq [(p+1)\int_0^1 u^p(t)dt]^{\frac{1}{p}}.$$

Lemma 3. *If $\alpha > -1$, $0 < p \leq q$ and u is nonnegative such that $u(t^{\frac{1}{1+\alpha}})$ is concave on $[0,1]$, then*

$$[(\alpha+1)(q+1)\int_0^1 u^q(t)t^\alpha dt]^{\frac{1}{q}} \leq [(\alpha+1)(p+1)\int_0^1 u^p(t)t^\alpha dt]^{\frac{1}{p}}. \qquad (2.1)$$

Proof. Using Lemma 2, we have

$$[(q+1)\int_0^1 u^q(t^{\frac{1}{1+\alpha}})dt]^{\frac{1}{q}} \leq [(p+1)\int_0^1 u^p(t^{\frac{1}{1+\alpha}})dt]^{\frac{1}{p}},$$

for $0 < p \leq q$.

With the change of variable $x = t^{\frac{1}{1+\alpha}}$, we have the desired inequality (2.1).

Remark. Lemma 2 is the special case of Lemma 3 when $\alpha = 0$.

Theorem 1. *Let $\alpha > -1$; $p, q > 0$. If u and v are nonnegative such that $u(t^{\frac{1}{1+\alpha}})$ and $v(t^{\frac{1}{1+\alpha}})$ are concave on $[0,1]$, then we have*

$$\int_0^1 u^p(t)v^q(t)dt \geq [\frac{2}{(p+1)(q+1)} - \frac{1}{\sqrt{(2p+1)(2q+1)}}](A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}}$$
$$\cdot(\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1}(\int_0^1 [u(t)]^A t^\alpha dt)^{\frac{p}{A}}(\int_0^1 [v(t)]^B t^\alpha dt)^{\frac{q}{B}}, \quad (3.1)$$

for $p \le A \le 2p$, $q \le B \le 2q$; and

$$\int_0^1 u^p(t)v^q(t)t^\alpha dt \ge [\frac{2(A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}}}{(p+1)(q+1)} - 1](\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1} \cdot (\int_0^1 [u(t)]^A t^\alpha dt)^{\frac{p}{A}} (\int_0^1 [v(t)]^B t^\alpha dt)^{\frac{q}{B}}, \tag{3.2}$$

for $A \ge 2p$, $B \ge 2q$.

Proof. If we replace u and v by $u^p(t)t^{\frac{\alpha}{2}}$ and $v^q(t)t^{\frac{\alpha}{2}}$, respectively, in Lemma 1, and set $w(t) = t^{\frac{\alpha}{2}}$, then we have the inequality

$$\frac{1}{2}[(\int_0^1 u^{2p}(t)t^\alpha dt)^{\frac{1}{2}}(\int_0^1 v^{2q}(t)t^\alpha dt)^{\frac{1}{2}} + \int_0^1 u^p(t)v^q(t)t^\alpha dt] \ge (\alpha+1)(\int_0^1 u^p(t)t^\alpha dt)(\int_0^1 v^q(t)t^\alpha dt). \tag{3.3}$$

Since $0 < p \le A$, $0 < q \le B$, using Lemma 3 to the right hand side of (3.3), we have

$$\int_0^1 u^p(t)v^q(t)t^\alpha dt \ge \Big[\frac{2(A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}}}{(p+1)(q+1)}(\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1} \cdot (\int_0^1 u^A(t)t^\alpha dt)^{\frac{p}{A}} \cdot (\int_0^1 v^B(t)t^\alpha dt)^{\frac{q}{B}}\Big] - (\int_0^1 u^{2p}(t)t^\alpha dt)^{\frac{1}{2}}(\int_0^1 v^{2q}(t)t^\alpha dt)^{\frac{1}{2}}. \tag{3.4}$$

Now if $A \le 2p$, $B \le 2q$ using Lemma 3 again to the last term of (3.4) , we have

$$(\int_0^1 u^{2p}(t)t^\alpha dt)^{\frac{1}{2}}(\int_0^1 v^{2q}(t)t^\alpha dt)^{\frac{1}{2}} \le \Big(\frac{(A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}}(\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1}}{\sqrt{(2p+1)(2q+1)}}\Big) \cdot (\int_0^1 u^A(t)t^\alpha dt)^{\frac{p}{A}}(\int_0^1 v^B(t)t^\alpha dt)^{\frac{q}{B}}. \tag{3.5}$$

The inequality (3.1) follows immediately from (3.4) and (3.5).

If $A \ge 2p$, $B \ge 2q$, an application of Hölder inequality yields

$$(\int_0^1 u^{2p}(t)t^\alpha dt)^{\frac{1}{2}}(\int_0^1 v^{2q}(t)t^\alpha dt)^{\frac{1}{2}} \le (\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1} \cdot (\int_0^1 u^A(t)t^\alpha dt)^{\frac{p}{A}}(\int_0^1 v^B(t)t^\alpha dt)^{\frac{q}{B}}. \tag{3.6}$$

The inequality (3.2) follows from (3.4) and (3.6).

Corollary 1. *Let $\alpha \geq 0$, $p, q > 0$. If u and v are nonnegative, concave and increasing on $[0,1]$, then (3.1) and (3.2) hold.*

Proof. Let $g(t) = t^{\frac{1}{1+\alpha}}$, $t \in [0,1]$. Then g is concave on $[0,1]$. If $t_1, t_2 \in [0,1]$, and let $f(t) = u(t^{\frac{1}{1+\alpha}})$, $t \in [0,1]$. Then

$$\begin{aligned} f(at_1 + bt_2) &= u(g(at_1 + bt_2)) \geq u(ag(t_1) + bg(t_2)) \\ &\geq au(g(t_1)) + bu(g(t_2)) \\ &= af(t_1) + bf(t_2). \end{aligned}$$

for all $a, b \in [0,1]$ with $a + b = 1$.

This shows that $u(t^{\frac{1}{1+\alpha}})$ is concave on $[0,1]$. Similarly, $v(t^{\frac{1}{1+\alpha}})$ is concave on $[0,1]$. The results then follows from Theorem 1.

Corollary 2. *Let $u(t)$ and $v(t)$ be nonnegative and concave on $[0,1]$ and $0 < p \leq A$, $0 < q \leq B$. Then we have*

$$\int_0^1 u^p(t)v^q(t)dt \geq \Big[\frac{2}{(p+1)(q+1)} - \frac{1}{\sqrt{(2p+1)(2q+1)}}\Big](A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}} \cdot \Big(\int_0^1 [u(t)]^A dt\Big)^{\frac{p}{A}} \Big(\int_0^1 [v(t)]^B dt\Big)^{\frac{q}{B}},$$

for $A \leq 2p$, $B \leq 2q$; and

$$\int_0^1 u^p(t)v^q(t)dt \geq \Big[\frac{2(A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}}}{(p+1)(q+1)} - 1\Big]\Big(\int_0^1 [u(t)]^A dt\Big)^{\frac{p}{A}} \Big(\int_0^1 [v(t)]^B dt\Big)^{\frac{q}{B}},$$

for $A \geq 2p$, $B \geq 2q$.

Proof. This is the case $\alpha = 0$ of Theorem 1.

Remarks.

1. Inequality (1.3) is the special case of Corollary 2 by taking $A = 2p$, $B = 2q$.
2. If we let $p = q = 1$, then Corollary 2 becomes :

$$\int_0^1 u(t)v(t)dt \geq \frac{(A+1)^{\frac{1}{A}}(B+1)^{\frac{1}{B}}}{6}\Big(\int_0^1 u^A(t)dt\Big)^{\frac{1}{A}}\Big(\int_0^1 v^B(t)dt\Big)^{\frac{1}{B}},$$

for $1 \leq A \leq 2$, $1 \leq B \leq 2$; and

$$\int_0^1 u(t)v(t)dt \geq \left[\frac{(A+1)^{\frac{1}{A}}(B+1)^{\frac{1}{B}}}{2} - 1\right]\left(\int_0^1 u^A(t)dt\right)^{\frac{1}{A}}\left(\int_0^1 v^B(t)dt\right)^{\frac{1}{B}},$$

for $A \geq 2$, $B \geq 2$.

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