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HERMITIAN SURFACES OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE II

TAKUJI SATO AND KOUEI SEKIGAWA

Abstract. The present paper is a continuation of our previous work [7]. We shall prove that a copmact Hermitian surface of pointwise positive constant holomorphic sectional curvature is biholomorphically equivalent to a complex projective surface.

§1. Introduction

An almost Hermitian manifold with integrable almost complex structure is called a Hermitian manifold. The holomorphic sectional curvature H of a Hermitian manifold $M = (M, J, g)$ can be regarded as a differentiable function on the unit tangent bundle $U(M)$ of M. A Hermitian manifold $M = (M, J, g)$ is said to be of pointwise constant holomorphic sectional curvature if the function H is constant along each fibre of $U(M)$. Especially, M is said to be of constant holomorphic sectional curvature if H is constant on the whole of $U(M)$. In the present paper, we shall deal only with the curvature tensor with respect to the Riemannian connection of M.

In the previous article [7], we have proved the following

Theorem A. *Let $M = (M, J, g)$ be a compact Hermitian surface of non-positive constant holomorphic sectional curvature. Then M is a Kähler surface.*

Theorem B. *Let $M = (M, J, g)$ be a compact Hermitian surface of positive*

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constant holomorphic sectional curvature. Then the Euler number $\chi(M)$ and the Chern number $c_1(M)^2$ are positive, and the Pontrjagin number $p_1(M)$ is non-negative (and hence, M is an algebraic surface with positive Euler number and non-negative signature).

We have also proved that the Pontrjagin number $p_1(M)$ of a compact Hermitian surface $M = (M, J, g)$ of pointwise constant holomorphic sectional curvature is non-negative, and $p_1(M)$ is equal to zero if and only if M is a locally conformal Kähler surface with $\tau = 3\tau^*$. The purpose of the present paper is to improve the above Theorem B. Namely, we shall prove the following

Theorem C. *Let $M = (M, J, g)$ be a compact Hermitian surface of pointwise positive constant holomorphic sectional curvature. Then M is biholomorphically equivalent to a complex projective surface $P^2(\mathbf{C})$.*

We adopt the same notational convention as in our previous paper [7].

§2. Preliminaries

Let $M = (M, J, g)$ be a Hermitian surface and $\Omega = (\Omega_{ij})$ the Kähler form of M defined by $\Omega_{ij} = g_{ik}J_j^k$. We denote by ∇, $R = (R_{ijk}{}^l)$, $\rho = (\rho_{ij})$ and τ, the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The Ricci $*$-tensor $\rho^* = (\rho^*_{ij})$ and the $*$-scalar curvature τ^* are given respectively by

$$\rho^*_{ij} = \frac{1}{2} J_j^s R_{isa}{}^b J_b^a, \tag{2.1}$$

$$\tau^* = g^{ij} \rho^*_{ij}. \tag{2.2}$$

We denote by $\omega = (\omega_i)$ the Lee form of M. The Lee form ω is given by $\omega = \delta\Omega \circ J$ and satisfies the following equalities:

$$J^{ij}\nabla_i\omega_j = 0, \tag{2.3}$$

$$2\nabla_i J_j^k = \omega_a J_j^a \delta_i^k - \omega_a J^{ka} g_{ij} - \omega_j J_i^k + \omega^k J_{ij}, \tag{2.4}$$

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2, \tag{2.5}$$

([7], [10]).

We denote by $\chi(M)$, $c_1(M)^2$, $c_2(M)$ and $p_1(M)$ the Euler number, the first Chern number, the second Chern number and the first Pontrjagin number of M, respectively. We note that $c_2(M)$ is equal to $\chi(M)$, and $\frac{1}{3}p_1(M)$ is equal to the Hirzebruch signature of M.

Now, let $\widetilde{\nabla}$ be the Hermitian connection (known also as the Chern connection) of M and $\widetilde{\Gamma} = (\widetilde{\Gamma}^i_{jk})$ the coefficients of the connection $\widetilde{\nabla}$ in each coordinate neighborhood of M. Then we have

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk} - \frac{1}{2}\omega_k\delta^i_j - \frac{1}{2}J^a_j\omega_a J^i_k + \frac{1}{2}g_{jk}\omega^i, \tag{2.6}$$

where $\Gamma = (\Gamma^i_{jk})$ are the coefficients of the Riemannian connection ∇ (cf. [10]). By (2.6), the curvature tensor $\widetilde{R} = (\widetilde{R}_{ijk}{}^l)$ of the connection $\widetilde{\nabla}$ is given by

$$\begin{aligned}\widetilde{R}_{ijk}{}^l =& R_{ijk}{}^l + \frac{1}{2}\{(\nabla_j\omega_k + \frac{1}{2}\omega_j\omega_k)\delta^l_i - (\nabla_i\omega_k + \frac{1}{2}\omega_i\omega_k)\delta^l_j \\ &+ (\nabla_i\omega^l + \frac{1}{2}\omega_i\omega^l)g_{jk} - (\nabla_j\omega^l + \frac{1}{2}\omega_j\omega^l)g_{ik}\} \\ &+ \frac{\|\omega\|^2}{4}(g_{ik}\delta^l_j - g_{jk}\delta^l_i - 2J_{ij}J^l_k) \\ &+ \frac{1}{2}J^l_k\{J^a_i(\nabla_j\omega_a + \omega_j\omega_a) - J^a_j(\nabla_i\omega_a + \omega_i\omega_a)\}. \end{aligned} \tag{2.7}$$

We denote by $\widetilde{\rho} = (\widetilde{\rho}_{ij})$ and $\widetilde{\rho}^* = (\widetilde{\rho}^*_{ij})$ the tensor fields on M of type (0,2) defined respectively by

$$\widetilde{\rho}_{ij} = \widetilde{R}_{kij}{}^k$$

and

$$\widetilde{\rho}^*_{ij} = \frac{1}{2}J^s_j\widetilde{R}_{isa}{}^bJ^a_b.$$

Furthermore, we put

$$\widetilde{\tau} = g^{ij}\widetilde{\rho}_{ij} \quad \text{and} \quad \widetilde{\tau}^* = g^{ij}\widetilde{\rho}^*_{ij}.$$

Then, by (2.7), we have easily

$$\widetilde{\rho}_{jk} = \rho_{jk} + \frac{1}{2}\{\nabla_j\omega_k - J_j^a J_k^b(\nabla_b\omega_a + \omega_b\omega_a) - (\delta\omega)g_{jk}\}, \tag{2.8}$$

$$\widetilde{\rho}^*_{jk} = \rho^*_{jk} - \frac{1}{2}(\nabla_j\omega_k + J_j^a J_k^b \nabla_b\omega_a) - \frac{3}{4}(\omega_j\omega_k + J_j^a J_k^b\omega_b\omega_a - \|\omega\|^2 g_{jk}). \tag{2.9}$$

From (2.5), (2.8) and (2.9), we have

$$\widetilde{\tau} = \tau^* + \frac{1}{2}\|\omega\|^2, \tag{2.10}$$

$$\widetilde{\tau}^* = \frac{1}{2}(\tau + \tau^*) + \|\omega\|^2 . \tag{2.11}$$

Let u and v be the two scalar curvatures of Hermitian geometry introduced in the work of A. Balas [3]. Then we have

$$\widetilde{\tau} = 2v \quad \text{and} \quad \widetilde{\tau}^* = 2u. \tag{2.12}$$

We now assume that M is of pointwise constant holomorphic sectional curvature $c = c(p)$. Then we have

$$\tau + 3\tau^* = 24c, \tag{2.13}$$

(cf. [7]).

§3. Proof of Theorem C

Let $M = (M, J, g)$ be a compact Hermitian surface of pointwise positive constant holomorphic sectional curvature. Then the characteristic numbers $\chi(M)$, $c_1(M)^2$ and $p_1(M)$ are given respectively by

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{12c^2 - \frac{1}{16}(\tau - \tau^*)^2 + \frac{1}{2}\tau^*\|\omega\|^2\}dM, \tag{3.1}$$

$$c_1(M)^2 = \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tau^*\|\omega\|^2 + \|d\omega\|^2\}dM, \tag{3.2}$$

$$p_1(M) = \frac{1}{32\pi^2}\int_M \{\frac{1}{12}(\tau - 3\tau^*)^2 + \|d\omega\|^2\}dM, \tag{3.3}$$

(cf. [7]).

Now, by (2.10), (2.12) and (2.13), we have

$$u + v = \frac{1}{2}(\tilde{\tau}^* + \tilde{\tau}) = \frac{1}{4}(\tau + 3\tau^*) + \frac{3}{4} \| \omega \|^2 = 6c + \frac{3}{4} \| \omega \|^2 > 0. \tag{3.4}$$

Thus, from (3.4), taking account of the result of A. Balas ([3], Theorem 1), we see that the plurigenera of M all vanish, that is, the Kodaira dimension of M is equal to –1. Thus, the Noether formula ([5]) is of the form

$$c_1(M)^2 + c_2(M) = 12(1-q), \qquad (q \geq 0), \tag{3.5}$$

where $q = q(M)$ is the irregularity of M. From (3.5), taking account of Theorem B, we have $q = 0$. Thus, (3.5) reduces to

$$c_1(M)^2 + c_2(M) = 12. \tag{3.6}$$

Referring to the well-known classification of compact complex surfaces (see, for example, [4], p. 415), we infer that M is rational, equivalently, obtained by successive blowing up's from a complex projective surface $P^2(\mathrm{C})$ or a (geometrically) ruled surface over a complex projective line $P^1(\mathrm{C})$.

Since $c_2(M) = \chi(M) > 0$ by Theorem B, Miyaoka's inequality is of the form

$$c_1(M)^2 \leq 3c_2(M) \tag{3.7}$$

([5], [6]). By (3.6) and (3.7), we have

$$c_2(M) \geq 3. \tag{3.8}$$

Furthermore, by the Wu's theorem ([11], Theorem 10, p. 74) and Theorem B, we have

$$c_1(M)^2 \geq 2c_2(M). \tag{3.9}$$

So, by (3.6), (3.8) and (3.9), we have

$$c_2(M) = 3 \quad \text{or} \quad 4. \tag{3.10}$$

We assume that $c_2(M) = 4$. Then, by (3.6), we get $c_1(M)^2 = 8$. Thus, from the Wu's theorem, it follows immediately that $p_1(M) = 0$, and hence M is locally conformal Kähler surface with $\tau = 3\tau^*$ by virtue of (3.3). Then, by (2.13), we have

$$\tau = 12c > \tau^* = 4c.$$

Since M is simply connected, M is a globally conformal Kähler surface. Thus, there exists a differentiable function f on M such that $\omega = df$. Then the equality (2.5) reduces to

$$\tau - \tau^* = -2\Delta f + \| \operatorname{grad} f \|^2, \tag{3.11}$$

where $\Delta = -\delta d$ is the Laplace-Beltrami operator acting on differentiable functions on M. Let p_0 be a point of M such that $f(p_0) = \operatorname{Min}_{p \in M} f(p)$. Then, by (3.11), we see that $\tau \leq \tau^*$ at p_0. But this is a contradiction, and hence $c_2(M) = \chi(M) = 3$.

Thus, by (3.6) and the Wu's theorem, we have

$$c_1(M)^2 = 9 \qquad \text{and} \qquad p_1(M) = 3. \tag{3.12}$$

The second Betti number $b_2(M)$ of M is written as usual by $b_2(M) = b_+ + b_-$, and the signature of $M(= \frac{1}{3}p_1(M))$ is equal to $b_+ - b_-$. Since $\chi(M) = 2 + b_2(M) = 2 + b_+ + b_-$ and $b_+ - b_- = 1$ by (3.12), we have

$$b_+ = 1, \qquad b_- = 0. \tag{3.13}$$

Summing up the above arguments, we can conclude that M is biholomorphically equivalent to a complex projective surface $P^2(\mathbf{C})$. This completes the proof of Theorem C.

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Faculty of Technology, Kanazawa University, Kanazawa 920, Japan.

Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-21, Japan.