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ISOMETRIC IMMERSION OF MINIMAL SPHERICAL SUBMANIFOLD VIA THE SECOND STANDARD IMMERSION OF THE SPHERE*

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Abstract. Let M^n be a n-dimensional compact connected minimal submanifold of the unit sphere $S^{n+p}(1)$. In this paper we study the isometric immersion of M^n into $SM(n+p+1)$ via the second standard immersion of $S^{n+p}(1)$. We obtain some integral inequalities in terms of the spectrum of the Laplace operator of M^n, and find some restrictions on such immersions.

1. Introduction and Preliminaries

Let $\phi : M^n \to R^m$ be an isometric immersion of a compact connected n-dimensional Riemannian manifold M^n into a Euclidean m-space. Denote the spectrum of the Laplace-Beltrami operator $\triangle$ acting on differentiable functions in $C^\infty(M)$ by

$$Spec(M) = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \uparrow \infty\}$$

If we extend *triangle* to R^m-valued functions on M^n in a natural fashion, then we have the following spectral decomposition of ϕ (in L^2-sense)

$$\phi = \phi_0 + \sum_{t=1}^{\infty} \phi_t,$$

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$$\Delta\phi_t = \lambda_t\phi_t; \qquad \phi_t : M^n \to R^m,$$

where ϕ_0 is the center of mass of Min R^m. If there are exactly k-nonzero $\phi_t' s (t \geq 0)$ in the decomposition of ϕ, then we say that M^n is a k-type submanifold of R^m. Moreover, if $\phi_t = 0$ for all $t > q$, and $t < p$, $(1 \leq p \leq q)$ then M^n is called a k-type submanifold of R^m with order $[p, q]$, and the corresponding eigenvalues are called the k-type eigenvalues of M^n. If $p = q$, we simply say that M^n has order p. In this case, by a result of Takahashi [10], M^n is a minimal submanifold of a hypersphere $S^{m-1}(r)$ in R^m with $r^2 = \frac{n}{\lambda_p}$. For more detail concerning submanifolds of finite type, see [4].

Let $S^m(1)$ be the unit sphere in R^{m+1} with canonical inner product $<,>$. If we take each x in $S^m(1)$ as an $1 \times (m+1)$ row matrix, the second standard immersion f of $S^m(1)$ is given by

$$\begin{aligned} f &: S^m(1) \to SM(m+1) \\ f(x) &= x^t x \qquad \forall x \in S^m(1) \end{aligned}$$

where x^t is the transpose of x, and $SM(m+1)$ is the set of all $(m+1) \times (m+1)$ symmetric matrices over real numbers with a Riemannian metric g which is given by

$$g(P, Q) = \frac{1}{2} tr(P \cdot Q) \qquad \forall P, Q \in SM(m+1).$$

The second fundamental form σ of f satisfies the following condition:

$$\begin{aligned} g(\sigma(X,Y), \sigma(V,W)) &= 2 < X,Y >< V,W > + < X,V >< Y,W > \\ &+ < X,W >< Y,V >, \end{aligned}$$

If $\psi : M^n \to S^{n+p}(1)$ is a monimal immersion and $f : S^{n+p}(1) \to SM(n+p+1)$ is the second standard immersion of $S^{n+p}(1)$, then $\phi = f \circ \psi$ gives an isometric immersion of M^n into $SM(n+p+1)$. Suppose that the immersion ψ is full, i.e., $\psi(M^n)$ is not contained in any great hypersphere of $S^{n+p}(1)$. A. Ros [8] has proved that ϕ is of 2-type if and only if M^n is Einstein and $T = k <,>$, where T is a tenson field restricted on the normal bundle of ψ and $<,>$ is the canonical inner product on $S^{n+p}(1)$. It is known that ([8]) $f(S^{n+p}(1))$ is contained in a hypersphere $S^N(r)$ of $SM(n+p+1)$ with center $\frac{1}{n+p+1}I$ and radius $r = [\frac{n+p}{2(n+p+1)}]^{\frac{1}{2}}$ as a minimal submanifold, and $\phi(M^n)$ is mass symmetric in $S^N(r)$, i.e., $\phi_0 = \frac{1}{n+p+1}I$, where I is the $(n+p+1)\times(n+p+1)$ identity matrix. In this paper, we shall study some geometric inequalities involving these $k-type$ eigenvalues of M^n, $|h|^2$, and the scalar curvature of M^n, and find some restrictions on the manifold M^n, in order to have the isometric immersion of M^n into $SM(n+p+1)$.

2. Eigenvalue Inequalities

Let M^n be a n-dimensional compact connected minimal submanifold of the unit sphere $S^{n+p}(1)$, and f, ψ, ϕ, g, h and σ be defined as in the introduction.

Lemma 1. (Ros [8])

(1) $g(\phi-\phi_0, \phi-\phi_0) = \frac{n+p}{2(n+p+1)}$;

(2) $g(\phi, \Delta\phi) = n$;

(3) $g(\Delta\phi, \Delta\phi) = 2n(n+1)$;

(4) $g(\Delta^2\phi, \Delta\phi) = 4n(n+1)^2 + 4|h|^2$.

Lemma 2. *Let M^n be a full-immersed minimal submanifold of $S^{n+p}(1)$ with second fundamental form h, $\overline{H}$ be the mean curvature vector field of M^n in $SM(n+p+1)$ via the second standard immersion of $S^{n+p}(1)$. Then,*

(1) $|\overline{H}|^2 = \frac{2(n+1)}{n}$,

and (2) $|\Delta\overline{H}|^2 \le \frac{8(n+1)^3}{n} + \frac{8(n+1)}{n^2}|h|^2 + \frac{8}{n^2}|h|^4$,

with equality holds iff at most two of (h^α_{ij}) are nonzero which can be transformed simultaneously by an orthogonal matrix into a scalsr multiples of $\widetilde{A}$ and $\widetilde{B}$ respectively, where

$$\widetilde{A} = \begin{pmatrix} 1 & 1 & \\ 1 & 1 & 0 \\ & 0 & 0 \end{pmatrix}; \quad \widetilde{B} = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ & 0 & 0 \end{pmatrix}.$$

Proof. (1) can be obtained from Lemma 1 (3).

To prove (2), let us recall that (c.f.[8])

$$\Delta^2\phi = 2(n+1)\Delta\phi + 2\sum_{i,j,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}\alpha(\xi_\alpha,\xi_\beta) - 2\sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{ij}\sigma(E_j,E_k)$$

where $E_i, E_j, E_k, \ldots,$; are tangent vectors to M^n, and $\xi_\alpha, \xi_\beta, \xi_gamma, \ldots,$; are normal vectors to M^n; $i,j,k = 1,2,\ldots,n$; and $\alpha,\beta,\gamma = n+1,\ldots,n+p$. Let $H_\alpha = (h^\alpha_{ij})$. By a direct computation we have

$$|\Delta\overline{H}|^2 = \frac{1}{n^2}g(\Delta^2\phi,\Delta^2\phi),$$

and

$$\begin{aligned} g(\Delta^2\phi,\Delta^2\phi) =& g(\Delta^2\phi, 2(n+1)\Delta\phi + 2\sum_{i,j,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}\sigma(\xi_\alpha,\xi_\beta) \\ & - 2\sum_{i,j,k,\alpha,} h^\alpha_{ij}h^\alpha_{ij}\sigma(E_j,E_j)) \\ =& g(\Delta^2\phi, 2(n+1)\Delta\phi) + g(\Delta^2\phi, 2\sum_{i,j,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}\sigma(\xi_\alpha,\xi_\beta)) \\ & - g(\Delta^2\phi, 2\sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{ik}\sigma(E_j,E_k)) \\ =& \underbrace{2(n+1)g(\Delta^2\phi,\Delta\phi)}_{I} + \underbrace{2\sum_{i,j,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}g(\Delta^2\phi,\sigma(\xi_\alpha,\xi_\beta))}_{II} \\ & \underbrace{- 2\sum_{i,j,k,\alpha} h^\alpha_{ij}h^\alpha_{ik}g(\Delta^2\phi,\sigma(E_j,E_k))}_{III}. \end{aligned}$$

Here

$$\begin{aligned} I =& 2(n+1)[4n(n+1)^2 + 4|h|^2] \qquad \text{by Lemma 1 (4)}, \\ II =& 2\sum_{i,j,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}g(2(n+1)\Delta\phi + 2\sum_{s,t,\gamma,\tau} h^\gamma_{st}h^\tau_{st}\sigma(\xi_\gamma,\xi_\tau) \\ & - 2\sum_{s,t,u,\gamma} h^\gamma_{st}h^\gamma_{su}\sigma(E_t,E_u), \sigma(\xi_\alpha,\xi_\beta)) \\ =& 4(n+1)\sum_{i,j,\alpha,\beta} h^\alpha_{ij}h^\beta_{ij}g(-\sum_i \sigma(E_i,E_i), \sigma(\xi_\alpha,\xi_\beta)) \\ & + 4\sum_{i,j,s,t,\alpha,\beta,\gamma,\tau} h^\alpha_{ij}h^\beta_{ij}h^\gamma_{st}h^\tau_{st}g(\sigma(\xi_\gamma,\xi_\tau),\sigma(\xi_\alpha,\xi_\beta)) \end{aligned}$$

$$
\begin{aligned}
&-4\sum_{i,j,s,t,u,\alpha,\beta,\gamma} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\gamma}_{st}h^{\gamma}_{su}g(\sigma(E_t,E_u),\sigma(\xi_\alpha,\xi_\beta))\\
=&4(n+1)\sum_{i,j,\alpha,\beta} h^{\alpha}_{ij}h^{\beta}_{ij}(-2n\delta^{\beta}_{\alpha})\\
&+4\sum_{i,j,s,t,\alpha,\beta,\gamma,\tau} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\gamma}_{st}h^{\tau}_{st}(2\delta^{\tau}_{\gamma}\delta^{\beta}_{\alpha}+\delta^{\alpha}_{\gamma}\delta^{\beta}_{\tau}+\delta^{\alpha}_{\tau}\delta^{\beta}_{\gamma})\\
&-4\sum_{i,j,s,t,u,\alpha,\beta,\gamma} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\gamma}_{st}h^{\gamma}_{su}(2\delta^{u}_{t}\delta^{\beta}_{\alpha})\\
=&-8n(n+1)\mid h\mid^2+8\mid h\mid^4+4\sum_{i,j,s,t,\alpha,\beta} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\alpha}_{st}h^{\beta}_{st}\\
&+4\sum_{i,j,s,t,\alpha,\beta} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\alpha}_{st}h^{\beta}_{st}-8\mid h\mid^4\\
=&-8n(n+1)\mid h\mid^2+8\sum_{i,j,s,t,\alpha,\beta} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\alpha}_{st}h^{\beta}_{st}.
\end{aligned}
$$

By a similar calculation we also have

$$
III = 8n(n+1)\mid h\mid^2+8\sum_{i,j,k,s,\alpha,\beta} h^{\alpha}_{ij}h^{\alpha}_{ik}h^{\beta}_{sj}h^{\beta}_{sk}.
$$

Therefore,

$$
\begin{aligned}
g(\Delta^2\phi,\Delta^2\phi) =&I+II+III\\
=&2(n+1)[4n(n+1)^2+4\mid h\mid^2]\\
&+8[\sum_{i,j,s,t,\alpha,\beta} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\alpha}_{st}h^{\beta}_{st}+\sum_{i,j,k,s,\alpha,\beta} h^{\alpha}_{ij}h^{\alpha}_{ik}h^{\beta}_{sj}h^{\beta}_{sk}]\\
=&2(n+1)[4n(n+1)^2+4\mid h\mid^2]\\
&+8[\sum_{\alpha,\beta} Tr(H_\alpha H_\beta)^2+\sum_{\alpha,\beta} Tr(H_\alpha H_\beta H_\beta H_\alpha)]\\
=&2(n+1)[4n(n+1)^2+4\mid h\mid^2]\\
&+8[\sum_{\alpha,\beta} Tr(H_\alpha H_\beta)^2+\sum_{\alpha,\beta} Tr(H^2_\alpha H^2_\beta)]\\
\le&8n(n+1)^3+8(n+1)\mid h\mid^2+8\sum_{\alpha,\beta} TrH_\alpha rH^2_\beta\\
=&8n(n+1)^3+8(n+1)\mid h\mid^2+8\mid h\mid^4
\end{aligned}
$$

the inequality above and rest of the proof are due to the following lemma 3 which is a slight modification of a lemma in [5].

Q.E.D.

Lemma 3. *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$Tr(AB + BA)^2 \leq 2TrA^2 \cdot TrB^2$$

and the equality holds for nonzero matrices A and B if only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\widetilde{A}$ and $\widetilde{B}$ respectively, where

$$\widetilde{A} = \begin{pmatrix} 1 & 1 & \\ 1 & 1 & 0 \\ & 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ & 0 & 0 \end{pmatrix}.$$

Moreover, if A_1, A_2, and A_3 are $(n \times n)$-symmetric matrices and satisfy

$$Tr(A_\alpha A_\beta + A_\beta A\alpha)^2 = 2TrA_\alpha^2 A_\beta^2 \qquad 1 \leq \alpha,\ \beta \leq 3,$$

then at least one of the $A'_\alpha s$ must be zero.

Proof. We may assume that B is diagonal and denote the diagonal entries in B by $b_1, b_2, \ldots, b_n$. Then we have

$$\begin{aligned} Tr(AB + BA)^2 &= \sum_{i,j=1}^n a_{ij}^2 (b_i + b_j)^2 \leq 2 \sum_{i,j=1}^n a_{ij}^2 (b_i^2 + b_j^2) \\ &\leq 2 \sum_{i,j=1}^n a_{ij}^2 \sum_{k=1}^n b_k^2 = 2TrA^2 TrB^2. \qquad (*) \end{aligned}$$

Now, suppose that A and B are nonzero matrices and the equality holds. Then all equalities hold in $(*)$. From the second equality in $(*)$, it followss that

$$b_i = b_j \quad \text{if} \quad a_{ij} \neq 0, \qquad 1 \leq i,\ j \leq n.$$

Without loss of generality, we may assume that $a_{12} \neq 0$, then $b_1 = b_2$. From the third equality, we obtain that $b_3 = b_4 = \ldots = b_n = 0$. Since $B \neq 0$, it implies that $b_1 = b_2 \neq 0$, and it is easy to see that $a_{11} = a_{12} = a_{21} = a_{22} \neq 0$ and $a_{ij} = 0$ otherwise. If A_1, A_2, A_3 are three $(n \times n)$-symmetric matrices satisfy the equality in $(*)$, the argument above tell us that one of the them can be transformed to a scalar multiple of $\widetilde{A}$ as well as to a scalar multiple od $\widetilde{B}$, but $\widetilde{A}$ and $\widetilde{B}$ are not orthogonally equivalent, that one be zero. Q.E.D.

Theorem 1. *Let M^n be a full-immersed compact connected monomal submanifold of $S^{n+p}(1)$. Then*

$$\begin{aligned} 4\int_M \rho dv \leq & \{4n(n+1)^2 + n(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) \\ & - 2n(n+1)(\lambda_1+\lambda_2+\lambda_3) - \frac{\lambda_1\lambda_2\lambda_3}{2(n+pN1)} + 4n(n-1)\}vil(M), \end{aligned}$$

with equality holds if and only if either

(1) *M^n is 2-type in $SM(n+p+1)$ with order $[1,2]$, or $[1,3]$, or $[2,3]$,*

(2) *M^n is 3-type in $SM(n+p+1)$ with order $[1,3]$,*

where λ_1, λ_2, λ_3 are the first three nonzero eignvalues in Spec(M), and ρ is the scalar curvature of M^n.

Proof. Let

$$\begin{aligned} \Omega_1 &= \int_M g(\Delta\phi,\phi)dv - \lambda_1 \int_M g(\phi-\phi_0,\phi-\phi_0)dv; \\ \Omega_2 &= \int_M g(\Delta\phi,\Delta\phi)dv - \lambda_1 \int_M g(\Delta\phi,\phi)dv; \\ \Omega_3 &= \int_M g(\Delta^2\phi,\Delta\phi)dv - \lambda_1 \int_M g(\Delta\phi,\Delta\phi)dv; \\ a_i &= \int_M g(\phi_i,\phi_i)dv \qquad i = 1,2,\ldots,. \end{aligned}$$

By the orthogonality of the decomposition of ϕ, we know that

$$g(\phi_i,\phi_j) = 0 \qquad \text{if} \qquad i \neq j.$$

Therefore

$$\begin{aligned}\Omega_1 &= \sum_{t=2}^{\infty}(\lambda_t - \lambda_1)a_t;\\ \Omega_2 &= \sum_{t=2}^{\infty}\lambda_t(\lambda_t - \lambda_1)a_t;\\ \Omega_3 &= \sum_{t=2}^{\infty}\lambda_t^2(\lambda_t - \lambda_1)a_t.\end{aligned}$$

Hence,

$$\Omega_3 - (\lambda_2 + \lambda_3)\Omega_2 + \lambda_2\lambda_3\Omega_1 = \sum_{t=4}^{\infty}\Pi_{i=1}^{3}(\lambda_t - \lambda_i) \geq 0.$$

The theorem follows from Lemma 1 and the relation $\rho = n(n-1) - |h|^2$ for spherical minimal submanifold. Q.E.D.

Remark. (1) A similar inequality involving only λ_1, λ_2 was given by Ros ([8]). WHen $n = 2$, it was proved in [2].

(2) M^n can not be 1-type in $SM(n+p+1)$ via ϕ, we shall give the proof in next section.

In the case of 3-type, we have the following.

Theorem 2. *If M^n is a full-immersed compact connected minimal submanifold of $S^{n+p}(1)$, and has only three nonzero eigenvalues λ_{p_1}, λ_{p_2} and λ_{p_3} $(1 \leq p_1 \leq p_2 \leq p_3)$ in Spec(M). Then*

(1)
$$\begin{aligned}4\int_M \rho dv &= \{4n(n+1)^2 + n(\lambda_{p_1}\lambda_{p_2} + \lambda_{p_2}\lambda_{p_3} + \lambda_{p_3}\lambda_{p_1})\\ &\quad - 2n(n+1)(\lambda_{p_1} + \lambda_{p_2} + \lambda_{p_3}) - \frac{\lambda_{p_1}\lambda_{p_2}\lambda_{p_3}}{2(n+p+1)}\\ &\quad + 4n(n-1)\}vol(M),\end{aligned}$$

and

(2)
$$(\lambda_{p_1} + \lambda_{p_3}) - \frac{\lambda_{p_1}\lambda_{p_3}(n+1)}{2n(n+p+1)} \geq 2(n+1) \geq \max\left\{\begin{array}{l}(\lambda_{p_1} + \lambda_{p_2} - \frac{\lambda_{p_1}\lambda_{p_2}(n+1)}{2n(n+p+1)};\\ (\lambda_{p_2} + \lambda_{p_3}) - \frac{\lambda_{p_2}\lambda_{p_3}(n+1)}{2n(n+p+1)}.\end{array}\right\}$$

Proof. Only need to explain (2), that is a direct result by solving $a'_{p_i}s$ $i = 1, 2, 3$. From the proof of Theorem 1, it can obtained

$$a_{p_1} = \frac{vol(M)}{(\lambda_{p_1} - \lambda_{p_2})(\lambda_{p_1} - \lambda_{p_3})}\{\frac{\lambda_{p_2}\lambda_{p_3}(n+p)}{2(n+p+1)} + [2(n+1) - (\lambda_{p_2} + \lambda_{p_3})]n\};$$
$$a_{p_2} = \frac{vol(M)}{(\lambda_{p_2} - \lambda_{p_1})(\lambda_{p_2} - \lambda_{p_3})}\{\frac{\lambda_{p_1}\lambda_{p_3}(n+p)}{2(n+p+1)} + [2(n+1) - (\lambda_{p_1} + \lambda_{p_3})]n\};$$
$$a_{p_3} = \frac{vol(M)}{(\lambda_{p_3} - \lambda_{p_1})(\lambda_{p_3} - \lambda_{p_2})}\{\frac{\lambda_{p_1}\lambda_{p_2}(n+p)}{2(n+p+1)} + [2(n+1) - (\lambda_{p_1} + \lambda_{p_2})]n\}.$$

All $a'_{p_i}s \geq 0$, $i = 1, 2, 3$. Q.E.D.

Theorem 3. *Let M^n be a full-immersed compact connected minimal submanifold of $S^{n+p}(1)$ with second fundamental form h. Then*

$$\int_M |h|^2 \{\frac{8}{n^2} |h|^2 - [4\sum_{i=1}^{4}\lambda_i - \frac{8(n+1)}{n^2}]\} \geq \Gamma \cdot vol(M)$$

where

$$\Gamma = n \sum_{1\leq i<j<k\leq 4} \lambda_i\lambda_j\lambda_k + 4n(n+1)^2\sum_{i=1}^{4}\lambda_i - 2n(n+1)\sum_{1\leq i<j\leq 4}\lambda_i\lambda_j$$
$$- \frac{n+p}{2(n+p+1)}\Pi_{i=1}^{4}\lambda_i - \frac{8(n+1)(2n^2+2n+1)}{n^2}.$$

Corollary 1. *If $\Gamma \geq 0$, then*

$$|h|^2 < \frac{n^2}{2}\sum_{i=1}^{4}\lambda_i - (n+1) \text{ implies } |h| = 0$$

i.e., M^n is totally geodesic in $S^{n+p}(1)$.

Proof. Let Ω_1, Ω_2, Ω_3 be the same as in the proof of Theorem 1. Let

$$\Omega_4 = \int_M g(\Delta^2\phi, \Delta^2\phi)dv - \lambda_1 \int_M g(\Delta^2\phi, \Delta\phi)dv$$

and

$$\begin{aligned}\Omega_4^* &= \frac{8}{n^2}\int_M |h|^4\,dv + \frac{8(n+1)}{n^2}\int_M |h|^2\,dv + \frac{8(n+1)^3}{n}vol(M)\\ &\quad - \lambda_1\int_M g(\Delta^2,\Delta\phi)dv\\ &= \frac{8}{n^2}\int_M |h|^4\,dv + [\frac{8(n+1)}{n^2} - 4\lambda_1]\int_M |h|^2\,dv\\ &\quad + [\frac{8(n+1)^3}{n} - 4n(n+1)^2\lambda_1]vol(M).\end{aligned}$$

By a similar argument as we have used in the proof Theorem 1, we obtain that

$$\begin{aligned}&\Omega_4^* - (\lambda_2+\lambda_3+\lambda_4)\Omega_3 + (\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\Omega_2 - \lambda_2\lambda_3\lambda_4\Omega_1\\ &\geq \Omega_4 - (\lambda_2+\lambda_3+\lambda_4)\Omega_3 + (\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\Omega_2 - \lambda_2\lambda_3\lambda_4\Omega_1\\ &= \sum_{t=5}^{\infty}(\lambda_t-\lambda_4)(\lambda_t-\lambda_3)(\lambda_t-\lambda_2)(\lambda_t-\lambda_1)a_t \geq 0.\end{aligned}$$

Combine Lemma 1 and Lemma 2, by a long but direct computation, the theorem follows. Q.E.D.

Remark. The corollary gives a pinch theorem of Simons type ([9]) in terms of the spectrum of M^n, this shows a relation between the study of $Spec(M^n)$ and pinch condition on $|h|^2$.

3. Some Restrictions

In section 2, we always assume that M^n is full-immersed into $S^{n+p}(1)$, for such minimal submanifolds we have the following result.

Theorem 4. *If M^n is a full-immersed compact connected minimal submanifold of $S^{n+p}(1)$. Then M^n can not be immersed into any hypersphere of $SM(n+p+1)$ as a minimal submanifold via the standard immersion of $S^{n+p}(1)$.*

Proof. It is equivalent to show that M^n can not be 1-type submanifold of $SM(n+p+1)$ under the isometric immersion ϕ (defined in section 1). Suppose

M^n is of 1-type under ϕ with order k, then the Lemma 1 produces the following equalites:

$$\begin{aligned}
a_k &= \frac{n+p}{2(n+p+1)} \cdot vol(M); \\
\lambda_k a_k &= n \cdot vol(M); \\
\lambda_k^2 a_k &= 2n(n+1) \cdot vol(M); \\
\lambda_k^3 a_k &= 4n(n+1)^2 \cdot vol(M) + 4 \int_M |h|^2 \, dv.
\end{aligned}$$

It implies that $p = 0$ and $|h|^2 = 0$. Q.E.D.

After the first version of this paper was written, Prof. Dimitric kindly informed me that a much more general result 1-type submanifolds of $SM(n+p+1)$ was proved in his paper [6], and he also pointed out that a necessary condition for theorem 5 was missed in the original manuscript of this paper, I would like to express my thanks to him.

When $S^n \to S^{n+p}$ as a totally geodesic submanifold, it is known that S^n will be of 1-type in $SM(n+p+1)$ via the second standard immersion of s^{n+p} ([6]). However, S^n could be isometrically immersed into S^{n+p} as a minimal submanifold many different ways. Suggested by the classical Bernstein conjecture, S. S. Chern proposed the following "spherical Bernstein conjecture" in the international congress of mathematics at Nice: "If $S^{n-1}(1)$ is imbedded as a minimal hypersurface of $S^n(1)$, then it is an equator." The spherical Bernstein conjecture was disproved by W. Y, Hsiang in the dimension $n = 4, 5, 6, 7, 8, 10, 12$ and 14, he constructed infinitely many counterexamples in each of the above dimensions ([7]), later Tomtier gave counterexamples in even dimensions ([11]). For those full immersed hypersphers of spheres, according to theorem 4 they can not be of 1-type via the second immersion of the spheres. In addition, we also have the following general restrictions:

Theorem 5. *If $S^n(1)$ isometrically full immersed into $S^{n+1}(1)$ as a mass symmetric submanifold, then it can not be of finite-type in $SM(n+2)$ with order $[p, q]$ ($p \geq 2$) via the second standard immersion of $S^{n+1}(1)$.*

Proof. Let $f : S^{n+1}(1) \to SM(n+2)$ be the second standard immersion of $S^{n+1}(1)$, and $S^N(r)$ be the hypersphere of $SM(n+2)$ in which $f(S^{n+1}(1))$ is minimal, then by Takahashi ([10])

$$r^2 = \frac{n+1}{\lambda_2(S^{n+1}(1))} = \frac{n+1}{2(n+2)}.$$

Suppose $\psi : S^n(1) \to S^{n+1}(1)$ is an isometric full immersion such that

$$\phi = f \circ \psi : S^n(1) \to S^N(r) \subset SM(n+2)$$

is of finite type with order $[p, q]$, then $\dot{\phi}(S^n(1))$ is mass symmetric in $S^N(r)$, by a well known result in [4],

$$\lambda_2 S^n(1)) \leq \frac{n}{r^2};$$

That is

$$\begin{aligned} 2(n+1) &\leq \frac{2n(n+2)}{n+2}; \\ (n+2)(n+1) &\leq n(n+2); \\ n+1 &\leq n. \end{aligned}$$

This is a contradication. Q.E.D.

By a similar argument, we have

Corollary 2. *If $S^n(1)$ isometrically full immersed into $S^m(1)$ $(m > n)$ as a mass symmetric submanifold, then it can not be of finite type with order $[p, q]$ $(p \geq k \geq 2)$ in the k^{th} eigenspace of $S^m(1)$ via the k^{th} standard immersion of $S^m(1)$.*

Proof. Observe that $\lambda_k(S^n(1)) = k(k+n-1)$, and $S^m(1)$ is a minimal submanifold of some hypersphere $S^N(r)$ in R^{N+1} by its k^{th} standard immersion into R^{N+1}, rest of the argument identical with that of THeorem 5. Q.E.D.

Remark. Asimilar version of Theorem 5 for RP^n can be found in [4].

Let $\phi : \overline{M}^m \to R^{N+1}$ be an isometric immersion of a compact connected Riemannian manifold of 1-type, and $S^N(r)$ be the hypersphere of R^{N+1} in which

$\overline{M}^m$ is minimal. Suppose $\Delta\phi = \lambda_i\phi$, where Δ is the Laplace-Beltrami operator on $C^\infty(\overline{M})$ and λ_i is the i^{th} eigenvalue of Δ, then

$$r^2 = \lambda_i(\overline{M})$$

and $\phi(\overline{M}^m)$ is mass symmetric in $S^N(r)$, up to a congruence we may assume that $S^N(r)$ is centered at the origin of R^{N+1}. Denote such 1-type submanifold by $(\overline{M}^m, \lambda_i)$.

By Wallach [12], if $\overline{M} = \frac{G}{H}$ is a homogeneous Riemannian manifold with a G-invariant metric where G is a compact connected Lie group and the isotropy representation of the closed subgoup H is irreducible, then $\overline{M}$ is a such submanifold of 1-type. For instance, (S^m, m), $(S^m, 2(m+1))$, $(RP^m, 2(m+1))$, $(CP^m, 2(m+2))$, etc..

Let M^n be a submanifold of $(\overline{M}^m, \lambda_i)$ then M^n is mass symmetric in $\overline{M}^m$ if and only if it is mass symmetric in $S^N(r)$. It is known that all the minimal submanifolds of $(S^m(1), 2(m+1))$, and all the Kaehler submanifolds of $(CP^m, 2(m+2))$ are mass symmetric submanifolds. Theorem 5 inspires the following general result

Theorem 6. *Let M^n be an n-dimensional submanifold of $(\overline{M}^m, \lambda_i)$, it satisfies*

$$Ric(M^n) \geq (n-1)kg$$
$$k \geq \frac{1}{m}\lambda_i(\overline{M})$$

where g is the Riemannian metric on M^n, $k > 0$ is a constanr.

Then M^n can not be isometrically immersed into $\overline{M}^m$ as a mass symmetric submanifold unless $M^n = S^n(\frac{1}{\sqrt{k}})$, $\lambda_i(\overline{M}^m) = mk$. In particular, if $\overline{M}^m = S^m(\frac{1}{\sqrt{k}})$, $S^n(\frac{1}{\sqrt{k}})$ is the only mass symmetric submanifold (up to a congruence) of $\overline{M}^m$ with $Ric(M^n) \geq (n-1)kg$.

Proof. If M^n is a mass symmetric submanifold of $(\overline{M}^m, \lambda_i)$ and $S^N(r)$ is the sphere in which $\overline{M}^m$ is minimal, then by B. Y. Chen [4]

$$\lambda_1(M^n) \leq \frac{n}{r^2}$$

where $r^2 = \frac{m}{\lambda_i(\overline{M}^m)}$ (Takahashi [8]) that is

$$\lambda_1(M) \leq \frac{n\lambda_i(\overline{M}^m)}{m}$$

But from a result of Lichnerowicz (see [3])

$$\lambda_1(M^n) \geq nk$$

it follows that

$$\frac{n}{m}\lambda_i(\overline{M}^m) \geq nk$$

$$k \leq \frac{1}{m}\lambda_i(\overline{M}^m)$$

By the hypothese, the this is possible only if $mk = \lambda_i(\overline{M}^m)$ and $\lambda_1(M) = nk$. Therefore M^n must be $S^n(\frac{1}{\sqrt{k}})$ by obata [3]. Q.E.D.

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