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A SHORT NOTE ON TWO INEQUALITIES FOR SINE POLYNOMIALS

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Abstract. We present elementary proofs for

$$\sum_{\nu=1}^{n}(n+1-\nu)\ \sin(\nu x)\ >\ 0,$$

due to Lukács, and for

$$\sum_{\nu=1}^{n}\sin(\nu x)+\frac{1}{2}\ \sin((n+1)x)\ \geq\ 0, \qquad (*)$$

due to Fejér. Both inequalities are valid for $x\in(0,\pi)$ and $n=1,2,\ldots$. Furthermore we determine all cases of equality in $(*)$.

The subject of this paper are the trigonometric inequalities

$$\sum_{\nu=1}^{n}(n+1-\nu)\ \sin(\nu x)\ >\ 0 \qquad (1)$$

and

$$\sum_{\nu=1}^{n}\sin(\nu x)+\frac{1}{2}\ \sin((n+1)x)\ \geq\ 0, \qquad (2)$$

which are both valid for $x\in(0,\pi)$ and $n=1,2,\ldots$. Inequality (1) was discovered by F. Lukács who informed L. Fejér in a private communication about his result,

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and in 1928 Fejér [1, pp. 335-339] provided elegant (but not simple) proofs for (1) as well as (2) by showing that the sums on the left-hand side of (1) and (2) are coefficients of special power series with non-negative coefficients.

The aim of this note is to present elementary short proofs for inequalities (1) and (2). Furthermore we determine all cases of equality in (2); this result seems to be new.

Let $x \in (0, \pi)$ and $n \geq 1$. We define

$$t_n(x) = \sum_{\nu=1}^{n} \sin(\nu x);$$

then we obtain

$$\begin{aligned} t_n(x) + t_{n+1}(x) &= 2\sum_{\nu=1}^{n} \sin(\nu x) + \sin((n+1)x) \\ &= \frac{1}{\sin\frac{x}{2}}[\cos\frac{x}{2} - \cos((n+\frac{1}{2})x)] + \sin((n+1)x), \end{aligned}$$

and because of

$$\cos((n+\frac{1}{2})x) = \cos((n+1)x)\ \cos\frac{x}{2} + \sin((n+1)x)\ \sin\frac{x}{2}$$

we get

$$t_n(x) + t_{n+1}(x) = \cot\frac{x}{2}\ [1 - \cos((n+1)x)] \geq 0, \tag{3}$$

which proves inequality (2). Furthermore we conclude that the sign of equality holds in (2) if and only if $x = \dfrac{2\pi k}{n+1}$, $0 < k < \frac{n+1}{2}$ $(k \in z)$.

If we define

$$s_n(x) = \sum_{\nu=1}^{n} (n+1-\nu)\ \sin(\nu x),$$

then we have

$$s_n(x) = \sum_{\nu=1}^{n} t_\nu(x). \tag{4}$$

We consider two cases: If n is odd, say $n = 2m+1$ $(m \geq 0)$, then we obtain from (3) and (4):

$$s_{2m+1}(x) = t_1(x) + \sum_{\nu=1}^{m} [t_{2\nu}(x) + t_{2\nu+1}(x)] > 0.$$

And if n is even, say $n = 2m$ ($m \geq 1$), then we get

$$\begin{aligned} s_{2m}(x) &= \sum_{\nu=1}^{m}[t_{2\nu-1}(x) + t_{2\nu}(x)] \\ &\geq t_1(x) + t_2(x) = \cot\frac{x}{2}\,[1 - \cos(2x)] > 0. \end{aligned}$$

This establishes inequality (1).

Finally we mention two interesting results which are closely connected to Lukács' inequality. In 1954 A. H. Tureckiĭ [4] (see also [3, p. 252]) published a kind of converse of inequality (1):

$$\sum_{\nu=1}^{n}(n+1-\nu)\,|\sin(\nu x)| \leq \frac{(n+1)^2}{\pi}, \quad x \in \mathbb{R},\ n = 1,2,\ldots, \tag{5}$$

and in 1970 the following striking companion of (5) was proved by J. B. Kelly [2]:

$$\sum_{\nu=1}^{n}(-1)^{\nu+1}(n+1-\nu)\,|\sin(\nu x)| \geq 0, \quad x \in \mathbb{R},\ n = 1,2,\ldots.$$

References

[1] L. Fejér, "Einige Sätze, die sich auf das Vorzeichen einer ganzen rationalen Funktion beziehen; nebst Anwendungen dieser Sätze auf die Abschnitte und Abschnittsmittelwerte von ebenen und räumlichen harmonischen Entwicklungen und von beschränkten Potenzreihen", *Monatsh. Math. Phys.* 35 (1928), 305-344.

[2] J. B. Kelly, "Metric inequalities and symmetric differences", in: Inequalities II (0. Shisha, ed.), 193-212, Academic Press, New York, 1970.

[3] D. S. Mitrinović, "Analytic Inequalities", Springer-Verlag, New York, 1970.

[4] A. H. Tureckiĭ, "On a function deviating least from zero", (Russian), *Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat.* 16 (1954), 41-43.

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