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ON THE CONVERGENCE OF NEWTON-LIKE METHODS

IOANNIS K. ARGYROS AND FERENC SZIDAROVSZKY

Abstract. This paper examines conditions for the convergence of generalized Newton-like methods, and estimates the speed of convergence.

1. Introduction

In this paper we are concerned with the problem of finding conditions for the convergence of generalized Newton-like methods to a common fixed point x^* of mappings $f_k (k \geq 0)$ defined on a subset of a Banach space B. Such a problem is clearly important in numerical analysis since many applied problems reduce to locating fixed points x^* of such mappings. For example, iterations of the above type are extremely important in solving optimization problems as well as linear and nonlinear equations. A very important field of such applications can also be found in solving optimization problems in economy and solving nonlinear input-output systems (see ex. Fujimoto, [3], La Salle, [5], Okuguchi, [6], Okuguchi & Szidarovszky, [7], Ortega & Rheinboldt, [8], Polak, [9], Tishyadihigama, et al, [11]).

In particular, set $U(0,R) = \{x \in B/\|x\| \leq R\}$, consider the Newton-like iterates

$$x_{k+1} = x_k - A_k(x_k)^{-1}(f_{1k}(x_k) + f_{2k}(x_k)) \tag{1}$$

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for approximating a common fixed point $x^* = 0$ of equations

$$f_k(x) = f_{1k}(x) + f_{2k}(x) \qquad (k \geq 0). \tag{2}$$

Here, f_k, f_{1k}, f_{2k} are mappings defined on $U(0, R)$ with values in B, $A_k(\cdot)$ denote linear mappings which approximate the Frechet-derivative $f'_{1k}(x)$ of f_{1k} at $x \in U(0, R)$. The assumption $x^* = 0$, involves no loss of generality, since any solution x^* can be transformed into 0 by introducing the transformed mapping $q_k(x) = f_k(x + x^*) - x^*(k \geq 0)$.

2. Convergence Analysis

We assume that for all k, $k \geq 0$, and each fixed $r \in [0, R]$

(A) $A_k(0)^{-1}$ exists and for all x, $y \in U(0, r) \subseteq U(0, R)$,

$$\|A_k(0)^{-1}(A_k(x) - A_k(0))\| \leq w_0(\|x\|) + b, \tag{3}$$

$$\|A_k(0)^{-1}(f'_{1k}(tx) - A_k(x))\| \leq w(t\|x\|) + c, \; t \in [0, 1] \tag{4}$$

and

$$\|A_k(0)^{-1}(f_{2k}(x) - f_{2k}(y))\| \leq e(r)\|x - y\|, \tag{5}$$

where w_0, w and e are nondecreasing nonnegative functions and constants b, c satisfy $b \geq 0$, $c \geq 0$ and $b + c < 1$. Note that the differentiability of f_{2k} is not assumed.

The above conditions are more general than the ones considered by Argyros, [1], Dennis, [2], Kantorovich & Akilov, [4], Ortega & Rheinboldt, [8], Yamamoto & Chen, [12], Zabrejko & Nguen, [13] who treated the above problem when $f_k = f$, $k \geq 0$. They provided sufficient conditions for the convergence of Newton-like iterates (1) to 0 in this special case. We will proceed in a similar manner but for the more general case described above. Our results can be easily reduced to the ones obtained by the above authors when $f_k = f$, $k \geq 0$. However, we will leave that to the motivated reader.

Define now the functions

$$x^*(r) = \int_0^r w(s)ds + \int_0^r e(r)ds + (b + c - 1 + w_0(r))r, \tag{6}$$

and

$$g(r) = 1 - b - w_0(r) \text{ for all } r \in [0, R). \tag{7}$$

Introduce the difference equation

$$\rho_{k+1} = \rho_k + x^*(\rho_k)g(\rho_k)^{-1}(k \geq 0), \qquad \rho_o = R. \tag{8}$$

We can now formulate the main result:

Theorem. *Under Condition (A), assume there exist $x_0 \in B$, $R > 0$ such that 0 is the unique zero of function $x^*(r)$ given by (6) in $[0, R)$. Moreover, suppose $\|x_0\| \leq a \leq R$ and $x^*(R) \leq 0$.*

Then iterates generated by (1) are well defined for all $k \geq 0$, belong to $U(0, R)$ and converge to 0 with

$$\|x_k\| \leq \rho_k \qquad (k \geq 0), \tag{9}$$

where sequence ρ_k which is monotonically decreasing and coverges to 0 is given by (8).

Proof. We will first show that the sequence generated by (8) is monotonically decreasing and converges to 0. Since 0 is the unique zero of function $x^*(r)$ in $[0, R)$ and $x^*(R) \leq 0$,

$$x^*(r) < 0 \text{ for all } r \in [0, R). \tag{10}$$

By using (6) we get

$$0 \leq \int_0^r w(s)ds + \int_0^r e(r)ds < (1 - b - c - w_0(r))r$$

which implies that

$$g(r) > 0 \text{ for all } r \in [0, R). \tag{11}$$

Using relations (8), (10), (11) and finite induction, it is routine to show that sequence ρ_k is monotonically decreasing. Furthermore, iteration (8) can also be written as

$$\rho_{k+1} = [\int_0^{\rho_k} (w)(s)ds + e(\rho_k)\rho_k + c\rho_k]g(\rho_k)^{-1} \geq 0 \text{ for all } k \geq 0 \qquad (12)$$

which imply that

$$0 \leq \rho_{k+1} \leq \rho_k \qquad (k \geq 0).$$

Hence, there exists a $\rho^* \in [0, R)$ with $\rho_k \to \rho^*$ as $k \to \infty$. Note that from (8) and the uniqueness of 0 as a zero of $x^*(r)$ in $[0, R)$ we conclude that $\rho^* = 0$.

By induction on k we will show that (9) holds: For $k = 0$, (9) becomes $\|x_0\| \leq \rho_0 = R$, which is true since $a \leq R$ by hypothesis. Assume (9) holds for k. From (3) and (11) we get

$$\|A_k(0)^{-1}(A_k(x_k) - A_k(0))\| \leq w_0(\rho_k) + b < 1.$$

By the Banach lemma on invertible mappings $A_k(x_k)$ is invertible. By using identity

$$A_k(x_k) = A_k(0)[I + A_k(0)^{-1}(A_k(x_k) - A_k(0))],$$

we see that

$$\|A_k(x_k)^{-1}A_k(0)\| \leq g(\rho_k)^{-1} \qquad (k \geq 0). \qquad (13)$$

From (1) and the fact that 0 is a fixed point of equation (2) we obtain in turn

$$\begin{aligned}
x_{k+1} &= x_k - A(x_k)^{-1}f_k(x_k) \\
&= -A_k(x_k)^{-1}[f_k(x_k) - A_k(x_k)x_k] \\
&= -A_k(x_k)^{-1}[f_{1k}(x_k) + f_{2k}(x_k) - A_k(x_k)x_k - (f_{1k}(0) + f_{2k}(0))] \\
&= -A_k(x_k)^{-1}[(f_{1k}(x_k) - f_{1k}(0) - A_k(x_k)x_k) + (f_{2k}(x_k) - f_{2k}(0))] \\
&= -A_k(x_k)^{-1}[\int_0^1 f'_{1k}(tx_k)x_k dt - A_k(x_k)x_k + (f_{2k}(x_k) - f_{2k}(0))] \\
&= -[A_k(x_k)^{-1}A_k(0)]\Big\{\int_0^1 A_k(0)^{-1}[((f'_{1k}(tx_k) - A_k(x_k))x_k dt \\
&\quad + (f_{2k}(x_k) - f_{2k}(0))]\Big\}.
\end{aligned}$$

By taking norms in the above approximation and using the triangle inequality and (13) we obtain

$$\begin{aligned}
\|x_{k+1}\| &= \|[A_k(x_k)^{-1}A_k(0)]\{\int_0^1 A_k(0)^{-1}[((f'_{1k}(tx_k) - A_k(x_k))x_k)dt \\
&\quad + (f_{2k}(x_k) - f_{2k}(0))]\}\| \\
&\leq \|A_k(x_k)^{-1}A_k(0)\|\Big\{\|\int_0^1 A_k(0)^{-1}((f'_{1k}(tx_k) - A_k(x_k))x_k)dt\| \\
&\quad + \|A_k(0)^{-1}(f_{2k}(x_k) - f_{2k}(0))\|\Big\} \\
&\leq (\int [w(t\|x_k\|) + c + e(\|x_k\|)]\,\|x_k\|dt)g(\rho_k)^{-1} \\
&\leq (\int_0^{\rho_k} w(s)ds + \int_0^{\rho_k} e(\rho_k)ds + c\rho_k)g(\rho_k)^{-1} = \rho_{k+1},
\end{aligned} \tag{14}$$

since

$$x_k \in U(0, \|x_k\|) \subseteq U(0, \rho_k)$$

and

$$\|\int_0^1 A_k(0)^{-1}((f'_{1k}(tx_k) - A_k(x_k))x_k dt\| \leq \int_0^1 (w(t\|x_k\|) + c)\|x_k\|dt,$$
$$\|A_k(0)^{-1}(f_{2k}(x_k - f_{2k}(0))\| \leq e(\|x_k\|)\|x_k\|,$$

by (4) and (5) respectively.

Hence (9) holds for $k+1$. From relation (14) we conclude that $x_{k+1} \in U(0, R)$. Finally, by letting $k \to \infty$ in (14) we get $x_k \to 0$, which completes the proof.

In practical cases we can select $A_k(x_k)$ to be either $f'_{1k}(x_k)$ or $f'_{1k}(x_0)$ or $f'_{1k}(0)$ or $S_k(x_{k-1}, x_k)$ (secant mappings) or any other linear mapping satisfying relations (3)-(5).

Results for the above special methods can now easily follow.

References

[1] I.K. Argyros, "Newton-like methods under mild differentiability conditions with error analysis". *Bull. Austral. Math. Soc.* 37,2, (1987), 131-147.

[2] J.E. Dennis, "Toward a unified convergence theory for Newton-like methods". In: *Nonlinear functional analysis and applications* (L. B. Rall, ed.), 425-472, Academic Press, New York, 1971.

[3] T. Fujimoto, "Global asymptotic stability of nonlinear difference equations", *I. Econ. Letters*, 22, (1987), 247-250.

[4] L.V. Kantorovich, & G.P. Akilov, "Functional analysis in normed spaces". *Pergamon Press*, New York, 1964.

[5] J.P. LaSalle, "The stability and control of discrete processes". *Springer-Verlag*, New York, 1986.

[6] K. Okuguchi, "Mathematical foundation in economical analysis". *McGraw-Hill*, Tokyo, 1977 (in Japanese).

[7] K. Okuguchi, & F. Szidarovszky, "Theory of oligopoly with multi-product firms". *Springer -Verlag*, New York, 1990.

[8] J.M. Ortega, & W.C. Rheinboldt,"Iterative solutions of nonlinear equations in several variables". *Academic Press*, New York, 1970.

[9] E. Polak, "Computational methods in optimization: A unified approach". *Academic Press*, New York, 1971.

[10] W.C. Rheinboldt, "A unified convergence theory for a class of iterative processes". *SIAM J. Num. Anal.* 5,(1968), 42-63.

[11] S. Tishyadhigama, E. Polak, & R. Klessig, "A comparative study of several convergence conditions for algorithms modeled by point-to-set maps." *Mathematical Programming Study*, 10, (1979), 172-190.

[12] T. Yamamoto, & X. Chen, "Convergence domains of certain iterative methods for solving nonlinear equations". *Numer. Funct. Anal. & Optimiz.* 10, (1 & 2), (1989), 34-48.

[13] P.P Zabrejko, & D.F. Nguen, "The majorant method in the theory of Newton-Kantorovich approximations and the Ptak-error estimates". *Numer. Funct. Anal. & Optimiz.* 9, (1987), 671-684.

Department of Mathematics, Cameron University, Lawton, Ok 73505, U.S.A.

Department of Systems and Industrial Engineering, University of Arizona, Tucson, AZ 85721, U.S.A.