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ON SUCCESSIVE APPROXIMATIONS OF A GIVEN DIFFERENTIAL EQUATION

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In a paper [1], the successive approximate solution $y_n (n = 1, 2, 3, \ldots)$ of the differential equation

$$\frac{d^k y}{dx^k} - (x + ax^m)y = 0$$

has been discussed only in the case $k = 2$, and $m = 2, 3$. Expressions connecting the solutions y_n and y_{n+1} were given in this case. It is the object of the present paper to deal with the same problem but with higher order when $k = 3$, and m takes any positive value $(m \geq 2)$.

Differential equation is

$$\frac{d^3 y}{dx^3} - (x + ax^m)y = 0$$

For this porpose we collect together some results mainly for subsequent use.

1. Preliminary Results

Lemma 1. *The complete solution of the differential equation*

$$\frac{d^3 y}{dx^3} - xy = 0 \tag{1.1}$$

is

$$y = c_1 Ai\,x + c_2 Bi\,x + c_3 Ci\,x = \mu(x)$$

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where $Ai\ x$, $Bi\ x$, $Ci\ x$ *are the these Airy's integrals and* c_1, c_2, c_3, *are arbitaray constants. Hence*

$$\begin{aligned} Ai\,x &= \int_0^\infty e^{-t} \cos\, xt \cos h\, xt\, dt \\ Bi\ x &= \int_0^\infty e^{-t} \sin\, xt \sin h\, xt\, dt \\ Ci\ x &= \int_0^\infty e^{-t} \ (\sin\, xt \cos h\, xt \ + \ \cos\, xt \sin h\, xt)dt \quad \textit{see in } [1] \end{aligned}$$

Lemma 2. *Let the notation as in Lemma* 1, *and let* $u^{(n)}$ *denote the nth derivative of* u *with respect to* x. *Let further* H^{-1} *be the inverse operator of the operator*

$$\frac{d^3}{dx^3} - x$$

then

$$H^{-1}u^{(n)} = \frac{u^{(n+1)}}{n+1}$$

In [1], *the following relations have been required*

$$\begin{aligned} xu^{(n)} &= u^{(n+3)} - nu^{n-1} \\ x^2u^{(n)} &= u^{(n+6)} - (2n+3)u^{(n+2)} + (n-1)_2 u^{(n-2)} \\ x^3u^{(n)} &= u^{(n+9)} - (3n+9)u^{(n+5)} + \omega(n)u^{(n+1)} - (n-2)_3 u^{(n-3)} \end{aligned}$$

where

$$\begin{aligned} \omega(n) &= 3n^2 + 6n + 6, \qquad (n-1)_2 = n(n-1) \textit{ and} \\ (n-2)_3 &= n(n-1)(n-2) \end{aligned}$$

For the present situation, we require a similar expression for $x^m u^{(n)}$ *in general.*

Lemma 3.

$$\begin{aligned} x^m u^{(n)} &= a^{(n)}{}_{m,1} u^{(3m+n)} + a^{(n)}{}_{m,2} u^{(3m+n-4)} + \dots \\ &\quad + a^{(n)}{}_{m,i} u^{(3m+n-4i+4)} + \dots \end{aligned}$$

where

$$a^{(n)}{}_{m,1} = 1$$

$$a^{(n)}{}_{m,2} = -m[n + \frac{3}{2}(m-1)]$$

$$a^{(n)}{}_{m,i} = a^{(n)}{}_{m-i} - \{3m + n - 4i + 5\}a^{(n)}{}_{m-1,i-1}$$

This result is true for $m = 1,2,3$ and may be established in general by mathematical induction on m.

For the convenience of notation and typing we introduce the coefficients $\alpha(m,n,i)$ where

$$\alpha(m,n,i) = a^{(n)}{}_{m,i+1}$$

In this notation

$$\alpha(1,n,0) = 1, \ \alpha(1,n,1) = -n$$

$$\alpha(2,n,0) = 1, \ \alpha(2,n,1) = -(2n+3), \ \alpha(2,n,2) = (n-1)_2$$

$$\alpha(3,n,0) = 1, \ \alpha(3,n,1) = -(3n+9)$$

$$\alpha(3,n,2) = \omega(n), \ \alpha(3,n,3) = -(n-1)_3$$

$$\alpha(1,n,i) = 0 \quad \textit{for } m = 2, \ i \geq 2 \qquad \text{see[2]}$$

Moreover, by means of this notation Lemma 3. will be written.

Lemma 4.

$$x^{(m)}u^{(n)} = \alpha(m,n,0)u^{(3m+n)} + (m,n,1)u^{(3m+n-4)} + \dots$$

$$\sum_{i=0}^{k(m,n)} \alpha(m,n,i)u^{(3m+n-4i)}$$

where

$$\alpha(m,n,0) = 1$$

$$\alpha(m,n,1) = -m[n + 3/2(m-1)]$$

$$\alpha(m,n,i) = \alpha(m-1,n,i) - (3m+n-4i+4)\alpha(m-1,n,i-1)$$

and

$$k(m,n) = [(3m+n)|4]$$

2. Successive approximations of the differential equation

$$\frac{d^3y}{dx^3} - (x + ax^m)y = 0$$

In this section we apply the method of successive approximations (already used in the previous paper) for the differential equation

$$\frac{d^3y}{dx^3} - (x + ax^m)y = 0 \tag{2.1}$$

For this purpose we write (2.1) into the form

$$\frac{d^3y}{dx^3} - xy = ax^m y \tag{2.2}$$

If a is small enough, then for a first approximation, the above equation may be written

$$\frac{d3y}{dx3} - xy = 0$$

The solution of this is , by Lemma 1.,

$$y = \mu(x) = u.$$

This may be regardes as a first approximation y_1.

Thus $y_1 = u$.

For a second approximation, we put $y = y_1 = n$ in the R.H.S. of equation (2.2), thus having

$$\frac{d^3y}{dx^3} - xy = ax^m u \tag{2.3}$$

Thus by using Lemma 4. (with $n = 0$) for the R.H.S., equation (2.3) will be written

$$\frac{d3y}{dx3} - xy = a \sum_{i=0}^{k(m,0)} \alpha(m,0,i)u^{(3m-4i)} \tag{2.4}$$

The complementary function of this equation is evidently $y = u$, while the particular integral is by Lemma 2.,

$$aH^{-1}\left[\sum_{i=0}^{k(m,0)} \alpha(m,0,i)u^{(3m-4i)}\right]$$

$$= a \sum_{i=0}^{k(m,0)} \alpha(m,0,i)\frac{u^{(3m-4i+1)}}{(3m-4i+1)}$$

Thus the complete primitive of equation (2.4) given directly the approximate soluatio y_2, namely

$$y_2 = u + a \sum_{i=0}^{k(m,0)} \frac{\alpha(m,0,i)}{(3m-4i+1)} u^{(3m-4i+1)}$$

i.e.

$$y_2 = y_1 + a \sum_{i=0}^{k(m,0)} \frac{\alpha(m,0,i)}{(3m-4i+1)} u^{(3m-4i+1)}$$

For a third approximation, we put $y = y_2$ in the R.H.S. of (2.2), thus we have,

$$\frac{d^3y}{dx^3} - xy = ax^m u + a^2 \sum_{i=0}^{k(m,0)} \frac{\alpha(m,0,i)}{(3m-4i+1)} x^m u^{(3m-4i+1)} \tag{2.5}$$

Then using Lemma 4. twice for the R.H.S. of (2.5), we have

$$\begin{aligned}\frac{d^3y}{dx^3} - xy &= a \sum_{i=0}^{k(m,0,i)} \alpha(m,0,i) u^{(3m-4i)} \\ &+ a^2 \sum_{i=0}^{k(m,0)} \frac{\alpha(m,0,i)}{3m-4i+1} \sum_{j=0}^{k(m,3m-4i+1)} \alpha(m, 3m-4i+1, j) \\ &\cdot u^{(6m-4i+4j+1)}\end{aligned}$$

The complete solution of this equation gives the third approximation y_3 in the form

$$\begin{aligned}y_3 &= u + a \sum_{i=0}^{k(m,0)} \frac{\alpha(m,0,i)}{(3m-4i+1)} u^{(3m-4i+1)} \\ &+ a^2 \sum_{i=0}^{k(m,0)} \sum_{j=0}^{k(m,3m-4i+1)} \frac{\alpha(m,0,i)\alpha(m,3m-4i+1,j) \cdot u^{(6m-4i-4j+2)}}{(3m-4i+1)\,(6m-4i-4j+2)}\end{aligned}$$

i.e.

$$y_3 = y_2 + a^2 \sum_{i=0}^{k(m,0)} \sum_{j=0}^{k(m,3m-4i+1)} \frac{\alpha(m,0,i)\alpha(m,3m-4i+1,j) \cdot u^{(6m-4i-4j+2)}}{(3m-4i+1)\,(6m-4i-4j+2)}$$

Thus we have shown the following Lemma.

Lemma 5. *The first three approximate solutions of the differential equation*

$$\frac{d3y}{dx3} - (x + ax^m)y = 0$$

(a being small enough)

are

$$y_1 = u,$$

$$y_2 = y_1 + a \sum_{i=0}^{k(m,0)} \frac{\alpha(m,0,i)}{(3m-4i+1)} u^{(3m-4i+1)},$$

$$y_3 = y_2 + a^2 \sum_{i=0}^{k(m,0)} \sum_{j=0}^{k(m,3m-4i+1)} \frac{\alpha(m,0,i)\alpha(m,3m-4i+1,j) \cdot u^{(6m-4i-4j+2)}}{(3m-4i+1)(6m-4i-4j+2)}$$

In a similar way one can easily obtain the fourth approximation y_4. This will be given in the following Lemma.

Lemma 6.

$$y_4 = y_3 + a^3 \sum_{i=0}^{k} \sum_{i=0}^{k} \sum_{i=0}^{k} \frac{\alpha(m,0,i_1)\alpha(m,3m-4i_1+1,i_2)\alpha(m,6m-4i_1-4i_2+2,i_3)}{(3m-4i_1+1)(6m-4i_1-4i_2+2)(9m-4i_1-4i_2-4i_3+3)} \cdot u^{(9m-4i_1-4i_2-4i_3+3)}$$

where

$$k_1 = k(m,0)$$

$$k_2 = k(m,3m-4i_1+1)$$

and

$$k_3 = k(m,6m-4i_1-4i_2+2)$$

3. Simplification of the form of the approximate solutions:

The approximate solutions already obtained in the previous article may be simplified if we introduce the following symbols

$$\beta_1 = \frac{\alpha(m,0,i)}{3m-4i_1+1}$$
$$\beta_2 = \frac{\alpha(m,3m-4i_1+1,i_2)}{6m-4i_1-4i_2+2}$$
$$\beta_3 = \frac{\alpha(m,6m-4i_1-4i_2+2,i_3)}{9m-4i_1-4i_2-4i_3+3}$$

In this notation Lemma 5., 6 can be simplified.

Lemma 7. *The first four approximate solutions of the differential equation*

$$\frac{d^3y}{dx^3} - (x+ax^m)y = 0$$

(a small enough)

may be written

$$y_1 = u$$
$$y_2 = y_1 + a\sum_{i_1=0}^{k_1}\beta_1 u^{(3m-4i_1+1)},$$
$$y_3 = y_2 + a^2\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2}\beta_1\beta_2 u^{(6m-4i_1-4i_2+2)},$$
$$y_4 = y_3 + a^3\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2}\sum_{i_3=0}^{k_3}\beta_1\beta_2\beta_3 u^{(9m-4i_1-4i_2-4i_3+3)},$$

5. The case $m = 3$

In this case $m = 3$

$$\begin{aligned} \alpha(3,n,0) &= 1, \ \alpha(3,n,1) = -(3n+9) \\ \alpha(3,n,2) &= \omega(n), \ \alpha(3,n,3) = -n(n-1)(n-2), \\ \alpha(3,n,i) &= 0, \text{ for } i \geq 3 \qquad \text{see[1]}. \end{aligned}$$

Then, from theorem 1, we can apply also for case $m = 3$.

Aknowledgment

The author wishes to thank professor K. R. YACOUB, Department of Mathematics, University of Ain Shams, Egypt, for the valuable discussions in the present work which is quite helpful.

References

[1] K. R. YACOUB, "On the solutions of the differential equation", Acta F. R. N. Univ. Comen., *Mathematical* XXI-1968.

[2] K. R. YACOUB, "Expansion of three definite in power series", *Journal of Natural science and Mathematics*, VOL-VI, NO.2, 1966, 223-230.

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4. The approximate solutions y_n, y_{n+1}

In this section we express y_{n+1} in terms of y_n. For this purpose, we introduce the following:

$$
\begin{aligned}
k_1 &= k(m, 0), \\
k_{s+1} &= k(m, 3sm - 4(i_1 + i_2 + \ldots + i_s) + s); \\
&\quad S = 1, 2, 3, \ldots, n-1.
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_1 &= \frac{\alpha(m, 0, i)}{3m - 4i + 1} \\
\beta_{s+1} &= \frac{\alpha(m, 3sm - 4(i_1 + i_2 + \ldots + is) + is)}{3sm - 4(i_1 + i_2 + \ldots + i_s) + s} \\
&\quad \text{for } s = 1, 2, 3, \ldots, n-1.
\end{aligned}
$$

By means of this notation, we state the following.

Theorem 1. *Let y_n and y_{n+1} be the nth and $(n+1)$th approximate solutions of the differential equation*

$$\frac{d^3 y}{dx^3} - (x + ax^m)y = 0 \qquad (a \text{ small enough})$$

Then,

$$
\begin{aligned}
y_1 &= u, \\
Y_{n+1} &= y_n + a^n F_{m,n}(u) \qquad n \geq 1,
\end{aligned}
$$

where

$$F_{m,n}(u) = \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} \cdots \sum_{i_n=0}^{k_n} \{\prod_{s=1}^{n} \beta_s\} u^{(3sm - 4\sum_{s=1}^{n} i_s + s)}$$

This theorem is true for $n = 1, 2, 3$ and may be established in general by mathematical introduction on n. Although $F_{m,n}(u)$ is written in a compact form, yet such aform depends indeed on n processes of summation. However, $F_{m,n}(u)$ consist of afinite number of terms involving derivatives of u.

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and

$$\begin{aligned} \beta_1 &= \frac{\alpha(m,0,i)}{3m-4i+1} \\ \beta_{s+1} &= \frac{\alpha(m, 3sm - 4(i_1 + i_2 + \ldots + is) + is)}{3sm - 4(i_1 + i_2 + \ldots + i_s) + s} \\ &\quad \text{for } s = 1,2,3,\ldots,n-1. \end{aligned}$$

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