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FIXED POINTS AND BEST APPROXIMATIONS FOR MEASURABLE MULTIFUNCTIONS WITH STOCHASTIC DOMAIN

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Abstract. In this paper we prove a best approximation and a random fixed point theorems for Hausdorff continuous multifunctions with stochastic domain. Our result extend several earlier ones existing in the literature. We also show that in Engl [3] some of the hypotheses can be weakened.

1. Introduction

In this note we prove a fixed point principle and a best approximation theorem analogous to Reich's results [10]. This is done for a class of continuous multifunctions with stochastic domain. Our results are general enough to incorporate earlier works on this subject, like those of Itoh [5], Engl [3], Sehgal-Waters [13], Sehgal-Singh [12], Papageorgiou [9], Lin [8], and Xu [15].

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. We will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed, (convex) }\}.$$

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and

$$P_{k(c)}(X) = \{A \subseteq X : \text{ nonempty, compact, (convex) }\}.$$

A multifunction $F : \Omega \to P_f(X)$ is said to be measurable, if for all $x \in X$ the $\mathbb{R}_+$-valued function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : z \in F(\omega)\}$ is measurable. Recall that a measurable function $f : \Omega \to X$ is said to be a "measurable selector" of $F(\cdot)$, if for all $\omega \in \Omega$, $f(\omega) \in F(\omega)$. It turns out (see Wagner [14], theorem 4.2), that this definition of measurability is equivalent to saying that there exist measurable selectors $f_n : \Omega \to X$ $n \geq 1$ of $F(\cdot)$, *s.t.* for all $\omega \in \Omega$ $F(\omega) = \overline{\{f_n(\omega) : n = 1, 2, 3, \ldots\}}$, or that for every $U \subseteq X$ nonempty open $F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$. Following schäl [11] and Engl [3], we will say that $F : \Omega \to P_f(X)$ is "separable" if it is measurable and there exists a countable set $D \subseteq X$ *s.t.* for all $\omega \in \Omega$, $\overline{D \cap F(\omega)} = F(\omega)$.

Let $F : \Omega \to P_f(X)$ be a measurable multifunction and let $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\}$ (the graph of $F(\cdot)$). We know (see Wagner [14], theorem 4.2) that $GrF \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. Then $T : GrF \to 2^X\{\emptyset\}$ is a measurable multifunction with stochastic domain $F(\cdot)$ if and only if for all $U \subseteq X$ open, $\{\omega \in \Omega : T(\omega, x) \cap U \neq \emptyset, x \in F(\omega)\} \in \Sigma$. We will say that $T(\cdot, \cdot)$ is an h-continuous measurable multifunction with stochastic domain $F(\cdot)$, if in addition for every $\omega \in \Omega$, the multifunction $x \to T(\omega, x)$ is h-continuous on $F(\omega)$ (see definition below).

On $P_f(X)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by setting

$$h(A, B) = max[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)]$$

where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ and $d(b, A) = \inf\{\|b - a\| : a \in A\}$. It is well known that $(P_f(X), h)$ is a complete metric space and $(P_{kc}(X), h)$ a closed (hence complete) subset of it. In fact, $P_{kc}(X), h)$ is also separable, while $(P_f(X), h)$ is not. If Y is a topological space, a multifunction $F : Y \to P_f(X)$ is said to be Hausdorff continuous (h-continuous) if and only if it is continuous as a map from Y into the metric space $(P_f(X), h)$.

Let Z be a set and denote by $P(Z)$ is a collection of subsets of Z. Denote by $\hat{N}$ the set of all infinite sequences of positive integers and by $\hat{N}_0$, the set of all finite sequences of positive integers. A set $A \subseteq Z$ is said to be obtained from $P(Z)$ by applying the Souslin operation, if there exists a map $k : \hat{N}_0 \to P(Z)$ *s.t.* $A = \bigcup_{r \in \hat{N}} \bigcap_{n=1}^{\infty} k(r|n)$, where $r|n$ denotes the first n-elements of $r \in \hat{N}$ (see Jacobs [6]). Note that the union in the Souslin speration is uncountable. So if $P(Z)$ is a σ-field, then A may be outside of $P(Z)$. If however $P(Z)$ is closed under the Souslin operation, then we say that $P(Z)$ is a Souslin family. For example every complete σ-field is a Souslin family (see Wagner [14]).

Let (Ω, Σ) be a measurable space, Y a separable metrizable space and Z a metrizable space. A function $f : \Omega \times Y \to Z$ is said to be a Caratheodory function, if for all $y \in Y$, $\omega \to f(\omega, y)$ is measurable and for all $\omega \in \Omega$, $y \to f(\omega, y)$ is continuous. It is well-known that such a function is jointly measurable; i.e. $(\omega, y) \to f(\omega, y)$ is $(\Sigma \times B(Y), B(Z))$-measurable.

3. Main Results

We start with an approximation result, which can be viewed as the stochastic version of a result originally obtained by Reich [10]. Our result also generalizes theorem 2 of Sehgal-Singh [12], where the multifunction had a deterministic domain.

Theorem 3.1. *If (Ω, Σ) is a measurable space with Σ a Souslin family, X is a separable Banach space, $K : \Omega \to P_{kc}(X)$ is a separable multifunction and $T : GrK \to P_{kc}(X)$ is an h-continuous, measurable multifunction with stochastic domain $K(\cdot)$, then there exists a measurable map $x : \Omega \to X$ s.t. for all $\omega \in \Omega$ $x(\omega) \in K(\omega)$ and $d(x(\omega), T(\omega, x(\omega))) = \delta(K(\omega), T(\omega, x(\omega))) =$* $\inf\{\|v - w\| : v \in K(\omega), w \in T(\omega, x(\omega))\}$.

Proof. Using corollary 3.1 of Kandilakis-Papageorgiou [7], we can find a multifunction $\hat{T} : \Omega \times X \to P_{kc}(X)$ *s.t.* $\omega \to \hat{T}(\omega, x)$ is measurable, $x \to \hat{T}(\omega, x)$

is h-continuous and $\hat{T}|_{GrK} = T$. Then consider the multifunction $H : \Omega \to 2^X$ defined by

$$H(\omega) = \{y \in K(\omega) : d(y, \hat{T}(\omega, y)) = \delta(K(\omega), \hat{T}(\omega, y))\}.$$

From lemma 1.6 of Reich [10], we know that for every $\omega \in \Omega$, $H(\omega) \neq \emptyset$.

Let $\varphi : \Omega \times X \to \mathbb{R}_+$ be defined by $\varphi(\omega, x) = d(x, \hat{T}(\omega, x))$. Clearly $\varphi(\cdot,\cdot)$ is a Caratheodory function. Also let $k_n : \Omega \to X$ $n \geq 1$ be measurable selectors of $K(\cdot)$ *s.t.* for all $\omega \in \Omega$, $K(\omega) = \overline{\{k_n(\omega) : n = 1, 2, 3, \ldots\}}$ (see section 2). Then we have:

$$\delta(K(\omega)\hat{T}(\omega, y)) = \inf_{n \geq 1} \inf_{z \in \hat{T}(\omega, y)} \|k_n(\omega) - z\|.$$

Set $\psi_n(\omega, y) = \inf_{z \in \hat{T}(\omega, y)} \|k_n(\omega) - z\|$. Theorem 6.1 of [7] tells us that for every $n \geq 1$, $\omega \to \psi_n(\omega, y)$ is measurable, while proposition 23, p. 120 of Aubin-Ekeland [1] tells us that for every $n \geq 1$, $y \to \psi_n(\omega, y)$ is continuous. So $\psi_n(\cdot,\cdot)$ $n \geq 1$ is a Caratheodory function, hence jointly measurable. Thus $\psi(\omega, y) = \inf_{n \geq 1} \psi_n(\omega, y)$ is a measurable function *s.t.* for all $\omega \in \Omega$, $\psi(\omega, \cdot)$ is upper semicontinuous (see for example Bertsekas-Shreve [2], lemma 7.14, p. 147). Let $\eta(\omega, y) = \varphi(\omega, y) - \psi(\omega, y)$. Then clearly $\eta(\cdot,\cdot)$ is jointly measurable, and for every $\omega \in \Omega$, $\eta(\omega, \cdot)$ is lower semicontinuous. Also note that for all $(\omega, y) \in \Omega \times X$, $\eta(\omega, y) \geq 0$. Observe that $H(\omega) = \{y \in K(\omega) : \eta(\omega, y) \leq 0\}$. Hence for all $\omega \in \Omega$, $H(\omega) \in P_k(X)$.

Set $L_0(\omega) = \{y \in X : \eta(\omega, y) \leq 0\} = \{y \in X : \eta(\omega, y) = 0\}$ and observe that

$$H(\omega) = K(\omega) \cap L_0(\omega).$$

Since $\eta(\cdot,\cdot)$ is jointly measurable, we have $GrL_0 = \{(\omega, y) \in \Omega \times X : \eta(\omega, y) = 0\} \in \Sigma \times B(X)$. Since by hypothesis Σ is Souslin family, from theorem 4.2(g) of Wagner [14], we get that $L_0(\cdot)$ is measurable $\Rightarrow H(\cdot)$ is measurable. Applying the Kuratowski-Ryll Nardzewski selection theorem (see Wagner [14]), we get $x : \Omega \to X$ measurable *s.t.* for all $\omega \in \Omega$, $x(\omega) \in H(\omega)$. Then $x(\omega) \in K(\omega)$ and $d(x(\omega), T(\omega, x(\omega))) = \delta(K(\omega), T(\omega, x(\omega)))$ for all $\omega \in \Omega$. Q.E.D.

The next result is a general fixed point principle, that incorporates all random fixed point theorems involving continuous multifunctions. In particular, it contains as special cases the fixed point theorem of Itoh [5], theorems 8, 13, 14 and 15 of Engl [3], theorem 2 of Sehgal-Waters [13], corollary 1 of Sehgal-Singh [12], theorem 6 of Papageorgiou [9], theorems 4 and 5 of Lin [8] and theorems 1, 2, 3 and 4 of Xu [15]. From the above works, only Engl [3], considered multifunctions with stochastic domain. However he assumed that there exists a σ-finite measure $\mu(\cdot)$ on (Ω, Σ), that $intK(\omega) \neq \emptyset \mu - a.e.$ and that the multifunction $\omega \rightarrow intK(\omega)$ admits a measurable selector $x_0(\cdot)$. Furthermore, his random fixed point satisfies $x(\omega) \in T(\omega, x(\omega)) \mu - a.e.$ and not for all $\omega \in \Omega$. Our result drops all the above extra hypotheses of Engl [3] and obtains a random fixed point for every $\omega \in \Omega$ by assuming that Σ is Souslin family. In addition, our proof is considerably simpler and shorter than that of Engl [3]. Finally in proposition 3.3, we show that the selector hypothesis on the multifunction $\omega \rightarrow intK(\omega)$ is superfluous, since it is automatically implied by the other hypotheses that Engl [3] made.

Theorem 3.2. *If (Ω, Σ) is a measurable space with Σ a Souslin family, X a separable Banach space, $K : \Omega \rightarrow P_f(X)$ a separable multifunction, $T : GrK \rightarrow P_{kc}(X)$ is an h-continuous measurable multifunction with stochastic domain $K(\cdot)$ and for every $\omega \in \Omega$ $H(\omega) = \{x \in K(\omega) : x \in T(\omega, x)\} \neq \emptyset$, then there exists a measurable map $x : \Omega \rightarrow X$ s.t. for all $\omega \in \Omega$ $x(\omega) \in K(\omega)$ and $x(\omega) \in T(\omega, x(\omega))$.*

Remark. So this result says that under the above hypotheses, "deterministic" solvability of the fixed point problem implies "stochastic" solvability of it.

Proof. Applying corollary 1.3 of [7], we get $\hat{T} : \Omega \times X \rightarrow P_{kc}(X)$ a multifunction *s.t.* $\omega \rightarrow \hat{T}(\omega, x)$ is measurable, $x \rightarrow \hat{T}(\omega, x)$ is h-continuous and $\hat{T}|_{GrK} = T$. Let $H : \Omega \rightarrow 2^X$ be defined by

$$H(\omega) = \{x \in K(\omega) : x \in \hat{T}(\omega, x)\}.$$

By hypothesis, for all $\omega \in \Omega$ $H(\omega) \neq \emptyset$ and it is easy to check using the h-continuity of $\hat{T}(\omega, \cdot)$, that for all $\omega \in \Omega$ $H(\omega) \in P_f(X)$. Let $\varphi : \Omega \times X \rightarrow \mathbb{R}_+$

be defined by $\varphi(\omega, x) = d(x, \hat{T}(\omega, x))$. Clearly $\varphi(\cdot, \cdot)$ is a Caratheodory function. Then note that

$$GrH \ = \ GrK \cap \{(\omega, x) \in \Omega \times X : \varphi(\omega, x) = 0\} \ \in \Sigma \times B(X).$$

Since Σ is a Souslin family, theorem 4.2 of Wagner [14], tells us that $H(\cdot)$ is measurable. So applying the Kuratowski-Ryll Nardzewski selection theorem, we get $x : \Omega \rightarrow X$ measurable *s.t.* for all $\omega \in \Omega$, $x(\omega) \in H(\omega)$. Clearly $x(\cdot)$ is the desired random fixed point for $T(\cdot, \cdot)$. Q.E.D.

Finally in the next proposition, we show that in theorem 8 of Engl [3], the hypothesis that there exists a measurable function $x_0 : \Omega \rightarrow X$ *s.t.* $x_0(\omega) \in intK(\omega)\mu - a.e.$ is superfluous.

Proposition 3.3. *If (Ω, Σ, μ) is a σ-finite measure space, X is a separable Banach space and $K : \Omega \rightarrow P_f(X)$ is a multifunction s.t. $GrK \in \Sigma \times B(X)$ and $intK(\omega) \neq \emptyset$ $\mu - a.e.$, then there exist $x_0 : \Omega \rightarrow X$ measurable functin s.t. $x_0(\omega) \in intK(\omega)\mu - a.e.$*

Proof. Observe that

$$GrintK(\cdot) \ = \ \{(\omega, x) \in \Omega \times X : d(x, bdK(\omega)) > 0\} \ \cap \ GrK$$

where $bdK(\omega)$ denotes the boundary of $K(\omega)$. Note that $bdF(\omega)$ may be empty. In that case as usual $d(x, bdK(\omega)) = +\infty$. But from theorem 4.6 of Himmelberg [4], we know that if $D = \{\omega \in \Omega : bdK(\omega) \neq \emptyset\}$, then $D \in \Sigma$ and $\omega \rightarrow bdK(\omega)$ is measurable on D. Hence $\{(\omega, x) \in \Omega \times X : d(x, bdK(\omega)) > 0\} \ \cap \ GrK =$ $[\{(\omega, x) \in D \times X : d(x, bdK(\omega)) > 0\} \cup (D^c \times X)] \ \cap \ Grk \in \Sigma \times B(X)$. (Note that $(\omega, x) \rightarrow d(x, F(\omega))$ is a Caratheodory function on $D \times X$, hence jointly measurable). Therefore $GrintK \in \Sigma \times B(X)$. Apply Aumann's selection theorem (see Wagner [14], to get $x_0 : \Omega \rightarrow X$ measurable *s.t.* $x_0(\omega) \in intK(\omega)\mu - a.e.$ Q.E.D.

Remark. Note that our hypothesis on $K(\cdot)$ is weaker than that of Engl [3] (theorem 8), since we only assume graph measurability of $K(\cdot)$ and not measurability of it. Recall that for a $P_f(X)$-valued multifunction measurability implies graph measurability.

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