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ITERATIVE CONSTRUCTION OF FIXED POINTS OF A DISSIPATIVE TYPE OPERATOR

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Let D be a nonempty closed convex subset of a Hilbert space, H, with inner product $\langle \cdot, \cdot \rangle$, and let $T : D \mapsto D$ satisfy the dissipative type condition

$$Re\langle T(x) - T(y), x - y \rangle \leq C\|x - y\|^2 \tag{1}$$

for some $C < 1$ and all $x, y \in D$. If T is also Lipschitz continuous on D, then T has precisely one fixed point $\bar{x} \in D$ [1]. Moreover, if $\{C_n\} \subset (0,1]$ satisfies the following conditions:

$$\lim_{n\to\infty} C_n = 0 \qquad \sum_{n=0}^{\infty} C_n = \infty \tag{2}$$

then the recursion

$$x_{n+1} = (1 - C_n)x_n + C_n T(x_n) \qquad x_0 \in D \tag{3}$$

will converge to $\bar{x}$ [2]. Rhoades [3] has shown that if D is a bounded interval of R^1 and T maps D into D, then $\{x_n\}$ will converge to some fixed point of T if T is merely continuous and $\{C_n\}$ satisfies (2). The process (3) and the more general processes of Mann [4] and Ishikawa [5] have been investigated for operators satisfing various conditions of the continuity and contractivity type on special Banach spaces, a summary of some contributions to this field is given in [3,6-8].

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In [16], Dunn introduced the weaker version of (1) namely,

$$Re\langle \xi - \bar{x}, x - \bar{x}\rangle \leq C\|x - \bar{x}\|^2 \tag{4}$$

for some $\bar{x} \in D$, $C < 1$, and for all $x \in D$, $\xi \in Tx$. Here T is set valued and no continuity restrictions of any kind are imposed on T, also D need not be closed or open. Instead, it is assumed that the range of T is bounded. Dunn showed that if $x \in T(x)$, i.e., if x is a fixed point of T then $x = \bar{x}$, so that T can have at most one fixed point. Moreover, if $\{x_n\}$ is a sequence in D satisfying

$$x_{n+1} = (1 - C_n)x_n + C_n\xi_n \tag{5}$$

where $\xi_n \in T(x_n)$, with $\{C_n\} \subset (0, 1]$ satisfying

$$\sum_{n=0}^{\infty} C_n = \infty \qquad \sum_{n=0}^{\infty} C_n^2 < \infty \tag{6}$$

then $\{x_n\}$ strongly converges to $\bar{x}$.

Recently, Chidume [17] extended the results of Dunn [16] from Hilbert spaces to the L_p spaces for $p \geq 2$, which has at least two disjoint sets of positive finite measure.

Our purpose in this paper is to extend all results of Chidume and Dunn to the general complex uniformly smooth Banach spaces and obtain a convergence rate in the L^p, e^p, W_m^p spaces for $1 < p < \infty$.

1. Definitions and Preliminary Results

Let X be an arbitrary Banach space with dim $X \geq 2$, the modulus of convexity $\delta_X(\epsilon)$, $0 < \epsilon \leq 2$, of X is defined by

$$\delta_X(\epsilon) = inf\{1 - \|x + y\|/2 : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| = \epsilon\}$$

X is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for every $\epsilon > 0$, and uniformly smooth if the dual space X^* is uniformly convex. The estimation of the modulus of convexity for the spaces L^p, e^p, W_m^p, $1 < p < \infty$ are [21]:

$$\delta_X(\epsilon) \geq (p - 1/16)\epsilon^2, \quad 1 < p \leq 2$$

$$\delta_X(\epsilon) \geq p^{-1}(\epsilon/2)^p, \qquad p \geq 2$$

For the Banach space X, we shall denote by J the normalized duality mapping from X to 2^{X^*} given by

$$J_x = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^*\rangle\}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. If X is uniformly smooth, then J is single-valued and is uniformly continuous on bounded set.

We define for positive t

$$\beta(t) = \sup\{(\|x+ty\|^2 - \|x\|^2)/t - 2Re\langle y, J(x)\rangle : \|x\| \leq 1, \|y\| \leq 1\}$$

Clearly $\beta : (0,\infty) \to [0,\infty)$ is nondecreasing, continuous and $\beta(ct) \leq c\beta(t)$ for $c \geq 1$. Also we have

Lemma 1. (*Refer* [18]) *If X is a uniformly smooth Banach space and $\beta(t)$ is defined as above, then $\lim_{t\to 0+}\beta(t) = 0$ and*

$$\|x+y\|^2 \leq \|x\|^2 + 2Re\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|) \tag{7}$$

for all $x, y \in X$.

Proof. The proof is same as S.Reich proved for a real uniformly smooth Banach space [18].

Lemmer 2. *Let β_n be a nonegative sequence satisfying*

$$\beta_{n+1} \leq (1-\delta_n)\beta_n + \sigma_n$$

with $\delta_n \in [0,1], \sum_{i=1}^{\infty}\delta_i = \infty$ and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty}\beta_n = 0$.

Proof. Since $\sigma_n = o(\delta_n)$, let $\sigma_n = \epsilon_n \cdot \delta_n$, and $\epsilon_n \to 0$. By a straightforward induction, one obtains

$$0 \leq \beta_{n+1} \leq \Pi_{j=k}^{n}(1-\delta_j)\beta_k + \sum_{j=k}^{n}[\delta_j \prod_{i=j+1}^{n}(1-\delta_i)]\epsilon_j \quad (*)$$

We have

$$\prod_{j=k}^{n}(1-\delta_j) \le e^{-\Sigma_{j=k}^{n}\delta_j} \to 0$$

and

$$\sum_{j=k}^{n}\delta_j \prod_{i=j+1}^{n}(1-\delta_i) \le 1 \quad \forall n,k$$

Given $\epsilon > 0$, pick k such that $\epsilon_j \le \epsilon$ for all $j \ge k$, from $(*)$ we have

$$0 \le \liminf \beta_n \le \limsup \beta_n \le \epsilon$$

Letting $\epsilon \to 0$, we obtain $\lim_{n\to\infty} \beta_n = 0$.

For the uniformly smooth Banach spaces, we consider a mapping T: $D \mapsto 2^D$ satisfying

$$Re\langle \xi - \bar{x}, J(x-\bar{x})\rangle \le C\|x-\bar{x}\|^2 \tag{8}$$

for some $\bar{x} \in D$, $C < 1$, and for all $x \in D$, $\xi \in T(x)$. Actually, (8) is a natural generalization to a Banach space of the (4) for a Hilbert space.

2. Main Results

Theorem 1. *Let D be a subset of a uniformly smooth Banach space and $T : D \mapsto 2^D$, $\bar{x} \in D$ satisfy (8). If x is a fixed point of T, then $x = \bar{x}$, thus T can have at most one fixed point.*

Moreover suppose the range of T is bounded and let $\{x_n\} \subset D$ be generated by (5) with $\{C_n\} \subset (0,1]$ satisfying (2) then $\{x_n\}$ strongly converges to $\bar{x}$.

Proof. Let x be a fixed point of T, i.e., $x \in T(x)$. The condition (8) yields

$$\langle x-\bar{x}, J(x-\bar{x})\rangle \le C\|x-\bar{x}\|^2$$

so that

$$\|x-\bar{x}\|^2 \le C\|x-\bar{x}\|^2.$$

Since $C < 1$, this gives $x - \bar{x}$.

Now set

$$\beta_n = \|x_n - \bar{x}\|^2$$

and

$$d = \sup\{\|\xi - \bar{x}\| : \xi \in T(x), x \in D\} \tag{9}$$

Because $C_n \to 0$, it is easy to show there exists an integer $N \geq 1$ such that when $n \geq N$

$$[1-(1-C)C_n]^2 + d^2 C_n \beta(C_n) \leq 1$$

Let $B = \max\{\beta_i : 1 \leq i \leq N, 1\}$. First we want to show $\beta_n \leq B^2$ and

$$\beta_{n+1} \leq [1-(1-C)C_n]^2 \beta_n + B^2 d^2 C_n \beta(C_n). \tag{10}$$

From (5), (7), (8) and (9), we have

$$\begin{aligned}\beta_{n+1} &= \|x_{n+1} - \bar{x}\|^2 = \|(1-C_n)(x_n - \bar{x}) + C_n(\xi_n - \bar{x})\|^2 \\ &\leq (1-C_n)^2\|x_n - \bar{x}\|^2 + 2C_n(1-C_n)Re\langle \xi_n - \bar{x}, J(x_n - \bar{x})\rangle \\ &\quad + \max\{\|x_n - \bar{x}\|, 1\} C_n \|\xi_n - \bar{x}\| \beta(C_n\|\xi_n - \bar{x}\|) \\ &\leq (1-C_n)^2\beta_n + 2C_n(1-C_n)C\beta_n + \max\{\|x_n - \bar{x}\|, 1\} d^2 C_n \beta(C_n) \\ &\leq [1-(1-C)C_n]^2\beta_n + \max\{\beta_n, 1\} d^2 C_n \beta(C_n)\end{aligned} \tag{11}$$

For $n \leq N$, by the definition of number B, we have $\beta_n \leq B^2$. For $n \geq N$, we apply induciton: assume $\beta_n \leq B^2$ then

$$\begin{aligned}\beta_{n+1} &\leq [1-(1-C)C_n]^2\beta_n + B^2 d^2 C_n \beta(C_n) \\ &\leq \{[1-(1-C)C_n]^2 + d^2 C_n \beta(C_n)\} B^2 \\ &\leq B^2\end{aligned}$$

So we obtain $\beta_n \leq B^2$ for all n and from (11) we get (10). Now apply Lemma 2 to inequality (10). We get that $\{x_n\}$ strongly converges to the fixed point $\bar{x}$.

Remark 1. If $Tx \neq \emptyset$ for $x \in D$, then by the axiom of choice T has a single-valued section, $T^* : D \mapsto D$, $T^*x \in Tx$. For any such section, and for D convex and $\{C_n\} \subset (0,1]$, the recursion

$$x_{n+1} = (1-C_n)x_n + C_n T^* x_n \qquad x_0 \in D$$

generates a sequence $\{x_n\} \subset D$.

Remark 2. Theorem 1 is a generalization of [17] in Banach spaces, and for $\{C_n\}$ we use weaker conditions (2) to replace stronger conditions (6) that Chidume used in $L_p (p \geq 2)$ spaces.

For the special uniformly smooth Banach spaces $X = L^p$, e^p, W^p_m, $1 < p < \infty$, we have the estimate [19]:

$$\beta(t) \leq Mt^{s-1}$$

Where $s = 2$ if $2 \leq p < \infty$, $s = p$ if $1 < p < 2$, and M is some constant. Then we are able to obtain a convergence rate in the setting of Theorem 1.

Theorem 2. *Let $X = L^p$, e^p, W^p_m and T, $\{x_n\}$, C_n be as in Theorem 1. Then we can find $\{C^*_n\}$ such that for corresponding sequence $\{x^*_n\}$ we have the estimate*

$$\|x^*_n - \bar{x}\| \leq O(1/n^{(s-1)/2}). \tag{12}$$

Where $s = 2$ if $2 \leq p < \infty$, $s = p$ if $1 < p < 2$. In a sense to be made precise below, the above estimate is the "best estimate" and the sequence $\{x^*_n\}$ is the "best sequence" in this class for each fixed value of the constant C in (8).

Proof. Let

$$\begin{aligned} C^*_n &= \frac{(1-C)^{1/(s-1)}}{1+n(1-C)^{s/(s-1)}} \\ x^*_{n+1} &= (1-C_n)x^*_n + C_n T x^*_n \\ \beta^*_n &= \|x^*_n - \bar{x}\|^2 \\ \alpha^*_{n+1} &= [1-(1-C)C^*_n]^s \alpha^*_n + C^{*s}_n \quad \alpha^*_1 \geq (MB^{*2}d^2)^{-1}\beta^*_1 \end{aligned}$$

Solve above equations we get

$$\alpha^*_n = \frac{\alpha^*_1}{[1+(n-1)\alpha^{*1/(s-1)}_1 (1-C)^{s/(s-1)}]^{s-1}}.$$

From (10) and the estimation of $\beta(t)$ in L_p spaces, we have $\beta^*_n \leq (MB^{*2}d^2)\alpha_n$, $n = 1, 2, \ldots$. Thus, we get (12). Now we turn to prove following claim. Claim:

for any sequence $\{C_n\}$, we get sequence

$$\alpha_{n+1} = [1-(1-C)C_n]^s \alpha_n + C_n \qquad \alpha_1 = \alpha_1^*.$$

Then we have $\alpha_n^* \le \alpha_n$, $n = 1, 2, 3 \ldots$. Consider the function

$$g(x) = [1-(1-C)x]^s \alpha_n^* + x^s.$$

Then

$$g'(x) = -(1-C)s(1-(1-C)x)^{s-1}\alpha_n^* + sx^{s-1}$$

Let $g'(x) = 0$, we get

$$x = \frac{\alpha_n^{*1/(s-1)}(1-C)^{1/(s-1)}}{1+\alpha_n^{*1/(s-1)}(1-C)^{s/(s-1)}}.$$

Combine this equation with previous equation, we have $x = C_n^*$, i.e. the minimum point is C_n^*. We apply induction: if $\alpha_k^* \le \alpha_k$ then

$$\begin{aligned}\alpha_{k+1}^* &= [1-(1-C)C_k^*]^s\alpha_k^* + C_k^{*s}\\ &\le [1-(1-C)C_k]^s\alpha_k^* + C_k^s\\ &\le [1-(1-C)C_k]^s\alpha_k + C_k^s \;=\; \alpha_{k+1}\end{aligned}$$

The proof is completed.

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